BOUNDARY CONDITIONS FOR INTERPOLATORY SUBDIVISION

by

HELENE NORHEIM SEMMERUD

THESIS

for the degree of

Master of Science

(Master i Anvendt matematikk og mekanikk)

Faculty of Mathematics and Natural Sciences
University of Oslo

June 2013

Det matematisk- naturvitenskapelige fakultet
Universitetet i Oslo

Contents

List of Figures

Abstract

The first part of this thesis will build up an overview of a small section of the field of subdivision, namely binary univariate schemes. Since most of the theory on subdivision is based on a uniform spacing of the initial control points, and dyadic refinement at the subsequent levels, we will try to uncover some of the results which are based on irregular spacing of the original points, and non-dyadic insertion of the new points. The four-point scheme is a special member of a family of interpolatory schemes first introduced by Dubuc and Deslauriers [6, 5]. The scheme was independently discovered and analysed by Dyn, Levin and Gregory [9], who also introduced the scheme with a tension parameter ω. Here the scheme will be a reappearing character throughout.

Secondly, we will introduce a general subdivision scheme for interpolating both the given function values and one or more derivative values at the end point. The formulation of the subdivision scheme will be given for most the general case, while smoothness-, and with it Hölder regularity, results will be given for *the cubic case*, where the first order end point derivative is assumed to be provided as well as function values. This cubic case can be view as a modification of the usual four-point scheme, in the sense that the subdivision rule is only changed at the first odd point. In definition of this scheme we will use a cubic osculatory interpolant to calculate the first new odd point, while we for the rest of the new odd points still will work with rules generated by ordinary Lagrangian cubic interpolation. The need for the two extra function values at the ends is replaced by one derivative value. In addition, we will investigate whether the given derivative value is interpolated by the limit curve. As in the paper by Floater [11], the main idea is to view the limit function as the limit of piecewise polynomials of odd degree and deduce smoothness results from this. An application of our new univariate scheme can be how to join two curves in a smooth fashion.

The cubic case is also generalized to a (bicubic) tensor product scheme. We do expect a similar smooth join of surfaces based on our bivariate scheme.

Numerical examples will be given, and for completeness we introduce the interpolation theory used in the derivations and analysis where appropriate.

Acknowledgements

I would like to start by expressing my gratitude to my supervisor Michael Floater for introducing me to the world of subdivision and for the guidance throughout the completion of this thesis. A special thanks goes to my parents for all the love and support. I would also like to thank my colleagues at B1002 for all the good times, with an extra thanks to Sasha, Jarle and Oda for valuable feedback. My appreciation goes to my dearest Leif Harald for always believing in me. Last, but not least, a warm thank you to my good friend Rebekka for all the smiles and proofreading.

Helene Norheim Semmerud
June 2013.

Introduction

Over the past century subdivision methods has grown to be an attractive way of generating smooth geometric shapes. They are used in 3D animation and general computer aided design for creating approximative or interpolatory curves, surfaces and volumes, see e.g [25, 22, 8]. Subdivision was used in e.g. Pixar's short film Geri's Game, the company have also released an open source subdivision library [18, 17]. The methods traditionally use only a finite number of initial data points, often referred to as control points. The resulting geometric objects of the methods are created through repeated refinement of the control points. The algorithms involved are usually computationally inexpensive and easy to implement due to their recursive nature and the fact that a new point is often constructed as simple averages of a finite number of neighbouring old points. The use of subdivision method is geometric design has also been appreciated due to fact that there exists methods for creating surfaces with arbitrary topology, see for example [27]. Since subdivision methods are local by construction, they are also popular for use in interactive editing of surfaces. Later it has also been discovered that subdivision fits into the framework of wavelet theory and multiresolution analysis [4, 23].

Surfaces which do not have an analytic representation are often called 'free form' surfaces, and subdivision has shown to be a an versatile method for designing such surfaces. Traditionally it has been focused on determining the smoothness in the interior of the surface, only enabling the definition of a closed surface as mentioned in [14]. The need for boundary conditions or boundary control of subdivision surfaces, enabling surfaces with boundaries, i.e. open surfaces, is also discussed in [14]. It is pointed out that representing the boundaries of an object is in several applications an important feature, both for visual accuracy with respect to real world appearance and for further computations. Boundary control can be useful when designing mechanical parts for instance.

Overview

In this subsection we will comment on the choices made throughout the completion of this work and the way the thesis is structured. In this thesis we do not consider approximative subdivision, only interpolatory, as we have tried to recover the effect of irregular parametrisation. We will focus mainly on one particular univariate interpolatory scheme, the four-point scheme. In standard univariate subdivision the focus has been mainly on equally spaced parametrisation of the points and one wonders what kind of advantages we might obtain by loosening the restrictions on the parametrisation. In [4] some interesting thoughts concerning this is presented. First of all, the immediate application, a designer might want to add new points at arbitrary positions independent of the initial parametrisation, and still maintain a certain degree of smoothness and visual fairness. Second of all, the irregular setting can be applied to wavelet theory and multiresolution analysis in connection to second generation wavelets [23]. In [15] they argue that the necessity of allowing an irregular parametrisation naturally arise, for example in the case of spline curves associated with irregularly spaces knots, and by arbitrary knot insertion.

The schemes discussed in this thesis will be univariate, with the extension to bivariate schemes through a tensor product construction. This in turn means that we consider only surfaces that admit a planar rectangular parametrisation.

In the first chapter we will introduce binary univariate subdivision. Further will we introduce the notions of convergence and higher order smoothness of the limit function. Two analysis methods are going to be discussed in detail and known smoothness results for the four-point scheme over various types of irregular parametrisations are presented.

In the next chapter we will present a general subdivision scheme for interpolating function values and derivatives up to some order s at the first interpolation point. These schemes can be viewed as a modification of the Dubuc-Deslauriers schemes presented in [5, 6]. The main focus of our work have been on a special case where a derivative is given at the first point as well as function values. This case is referred to as *the cubic case* and is presented in more detail. A tensor product extension of this cubic case is also introduced and discussed. A smooth join example is given in both the univariate and bivariate case. To end this chapter we review some related work done by Adi Levin in his doctorate dissertation [14] and an article by Cai [26] concerning the four-point scheme.

The third chapter includes a smoothness analysis of *the cubic case*, based on a technique developed in [11]. Finally we discuss the approximation order of this univariate scheme and include smoothness results for the tensor product extension of *the cubic case*.

1 Introduction to binary univariate subdivision

1.1 Introduction

In the following chapter we will introduce the concept of binary univariate subdivision. The general idea is quite simple. Given a finite sequence of values $\mathbf{f} = \{f_i\}, i \in \mathbb{Z}$, associated with a strictly increasing sequence of parameters $\mathbf{x} = \{x_i\}, i \in \mathbb{Z}$, we seek an approximation to the given data by repeated refinement. The values and parameters together are termed the initial control points and form the vertices in the initial control polygon. See Fig 1.1. Losely we state that a subdivision scheme is the process of iterative refinement, according to some rules, of the initial control polygon. This will be our informal definition until restated. We initialise the process by setting $\mathbf{f}_0 = \mathbf{f}$. Let j denote the current refinement level, and define $\mathbf{f}_j = \{f_{j,i}\}, j > 0, i \in \mathbb{Z}$ componentwise by

$$f_{j+1,i} = \sum_k m_{j,i-ak} f_{j,k}, \qquad \forall\ i\ \in\ \mathbb{Z}, \tag{1.1}$$

where $m_j = \{m_{j,i}\}$ denotes the j^{th} mask of the scheme. The subscript j indicates that the mask is dependent on the level of refinement, the scheme is thus termed non-stationary. a is called the arity of the scheme and relates the number of new control points to the old. If $a = 2$ the scheme is called binary and the number of control points gets approximately doubled at each iteration. In the rest of this chapter, and the thesis in general, we will assume that the subdivision schemes are all binary. The mask consists of a stencils, so for a binary scheme it consist of two stencils, governing the new even and odd points, respectively. The process can be split up into two parts, calculating the new odd and even points separately.

$$f_{j+1,2i} = \sum_k m_{j,2i-2k} f_{j,k}, \qquad \forall\ i\ \in\ \mathbb{Z}, \tag{1.2}$$

$$f_{j+1,2i+1} = \sum_k m_{j,2i+1-2k} f_{j,k}, \qquad \forall\ i\ \in\ \mathbb{Z}. \tag{1.3}$$

Example 1.1. *We start by introducing the binary scheme called Dubuc-Deslauriers scheme of order* 3*. This scheme commonly refered to as the four-point scheme, which we will revisit and properly introduce later on. We assume, for now, that the scheme is*

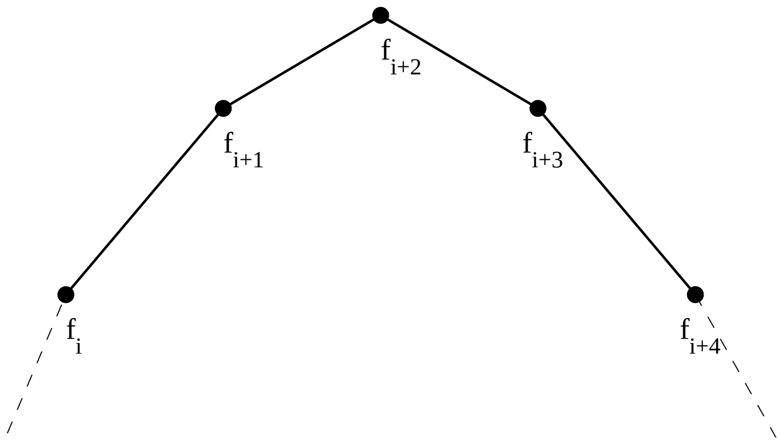

Figure 1.1: Control polygon

stationary. The mask is given as $m = \frac{1}{16}[-1, 0, 9, 16, 9, -1]$. *The even stencil is* $[0, 1, 0]$, *while the odd stencil is* $\frac{1}{16}[-1, 9, 9, -1]$, *and the scheme yields*

$$f_{j+1,2k} = f_{j,k}, \tag{1.4}$$

$$f_{j+1,2k+1} = -\frac{1}{16}(f_{j,k-1} + f_{j,k+2}) + \frac{9}{16}(f_{j,k} + f_{j,k+1}). \tag{1.5}$$

The odd stencil consists of the Lagrangian coefficients of the cubic interpolant at $-1, 0, 1, 2$ *evaluated at* $\frac{1}{2}$. *The scheme is interpolatory since all the old points are preserved, see (1.4). We notice that the mask is symmetric, making sure that the process act uniformly everywhere. The scheme is illustrated in Fig 1.2*

In this introduction we have chosen to focus on linear[1] schemes developed under some main assumptions. First of all: the scheme should be local. Locality ensures that a new point only depends on a finite number of neighbouring old points. This in turn means that the j^{th} mask is a finite set of real values. Locality also implies that a point on the limit curve[2] will depend on only a finite number of original points. In our review of Warrens article [24] this fact is illustrated for the four-point scheme, see Fig 1.4. Next we assume that the scheme is bounded. A scheme is termed bounded if all coefficients in the j^{th} mask are bounded by some constant M_s independent of j. In addition we assume that the scheme reproduces constants.We will then refer to the scheme as affine. This is another rather natural restriction, it means that the resulting shape is independent of the coordinate system. To summarize: we will only consider linear subdivision schemes that are *local, bounded, affine.* By the linear nature of (1.1) it is tempting to describe

[1] The scheme is termed linear when the new points depend linearly on the old.
[2] Which we will introduce in the next section.

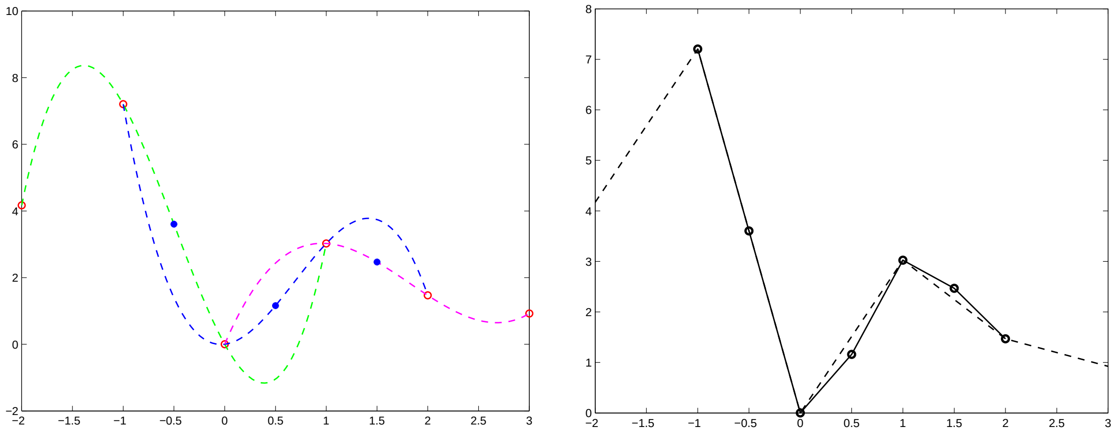

Figure 1.2: Illustration of the four-point scheme

the process through a matrix-vector product

$$\mathbf{f}_{j+1} = S_j \mathbf{f}_j \tag{1.6}$$

As stated in the introduction, the initial sequence of values is assumed to be finite, but we may view it as an inifinite yet finitely supported[3] sequence. Thus we also view the matrices as bi-infinite, indexed over $\mathbb{Z}^2$, to simplify index notation. Each j^{th} odd or even stencil forms the non-zero elements of the rows in S_j, alternating between even and odd. Each column contains a copy of the j^{th} mask. For (1.6) to be well-defined we have to ensure that each column contains only a finite set of non-zero values[24]. But this is easily verified, because the mask was restricted to a finite set by locality. So far we have stated that the mask may vary from level to level resulting in a non-stationary scheme, but it is worth mentioning that the mask may vary from one spatial position to another. The scheme is then termed non-uniform., and with the notation we now introduce we see that we get another mask index k, which shows that the masks, and stensils, may vary depending of parametrisation of the control points. We can now reformulate our definition of a subdivision scheme according to [15] more precisely;

Definition 1.1 (Subdivision scheme)**.** *A subdivision scheme* $\mathbf{S}$ *is an infinite sequence of matrices* $\mathbf{S} = \{S_j\}_{j\,\geq\,0}$ *where* $S_j := (S_{j,l,k})_{k,l\in\mathbb{Z}}$

By the definition above, the scheme is local meaning that there exists a number n_S independent of j such that

$$S_{j,l,k} \neq 0 \Rightarrow |l - 2k| \ \leq n_S \ \forall j \geq 0 \text{ and any } l, k \ \in \ \mathbb{Z}$$

[3]Only a finite subset of the index set is such that $f_i \neq 0$

The implication above can be split up into two, concerning the columns and rows separately.

$$2k - n_S \leq l \leq 2k + n_S,$$
$$\left\lceil \frac{l - n_S}{2} \right\rceil \leq k \leq \left\lfloor \frac{n_S + l}{2} \right\rfloor,$$

which in turn means that each column has of at most $2n_S + 1$ non-zero values, while each row has of at most $n_S + 1$ non-zero values. Then the rule (1.1) takes the form of the finite sum

$$f_{j+1,l} = \sum_{k=\lceil \frac{l-n_S}{2} \rceil}^{\lfloor \frac{n_S+l}{2} \rfloor} S_{j,l,k} f_{j,k} \tag{1.7}$$

A subdivision scheme is termed bounded if there exists a number $M_s < +\infty$ such that, for all $j \geq 0$,

$$\|S_j\| := \sup\{|S_{j,l,k}| \mid l, k \ \in \mathbb{Z}\} \leq M_S \tag{1.8}$$

A direct consequence of this last assumption is that if we start with a bounded initial sequence $\mathbf{f}_0$, each consecutive $\mathbf{f}_j$ will also be bounded.

Example 1.2. *Let us revisit the stationary four-point scheme. In this case, the bi-infinite subdivision matrix takes the following form*

$$S = \begin{pmatrix} \ddots & & & & & & & & \\ \dots & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & \dots \\ & & & & & & & & \ddots \end{pmatrix} \tag{1.9}$$

Here we see that $n_S = 3$, while $M_S = 1$

By following the notations and results from [15] we have now induced two properties on the scheme that are independent of the parametrisation of $\mathbf{f}_j$ at the different refinement levels, namely locality and boundedness. We emphasise that this is, analogous to the approach in [15], to introduce some global assumptions on the subdivision scheme, and later find out what properties are dependent of the parametrisation of $\mathbf{f}_j$. Another important property of local and bounded subdivision scheme is the existence of an associated bounded difference scheme, which is also independent of the parametrisation, and allows us to express the difference of control points across refinement levels.

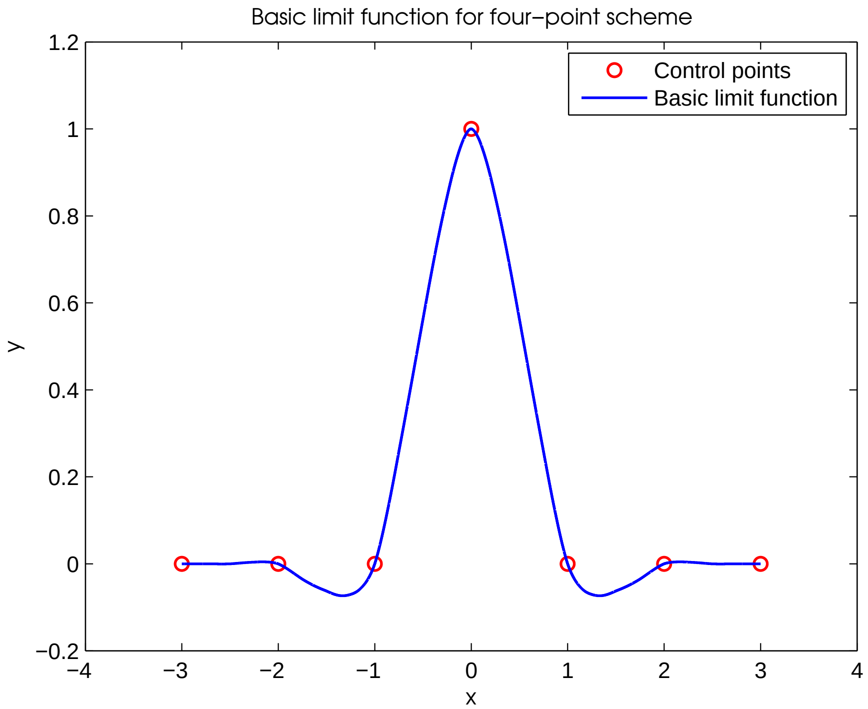

Figure 1.3: Basic limit function for the four-point scheme

Definition 1.2. *(Difference scheme associated with* **S** *[15]) Let* **S** *be a local and bounded subdivision scheme. Then the scheme* $\mathbf{D} = D_j,\ j \geq 0$ *defined by*

$$D_{j,k,l} = \sum_{i \geq l+1} (S_{j,k+1,i} - S_{j,k,i}), \qquad j \geq 0, \qquad k, l \in \mathbb{Z} \tag{1.10}$$

is called the difference scheme associated with **S**.

The difference scheme is also bounded since **S** is bounded and satisfies the *commutation formula*

$$\Delta S_j = D_j \Delta, \qquad j \geq 0 \tag{1.11}$$

where Δ, the difference operator, is the banded matrix with 1 on the main diagonal and -1 on the upper sub-diagonal. Since we assume that our scheme is affine we can deduce that our difference scheme is local as well [15]. Moreover, it can be shown that

$$\Delta \mathbf{f}_{j+1} = D_j \Delta \mathbf{f}_j \tag{1.12}$$

where $\Delta \mathbf{f}_{j+1} = (f_{j,k+1} - f_{j,k})_{k \in \mathbb{Z}}$. For the control polygon on level j, we can defined the affine function $g_j : \mathbb{R} \mapsto \mathbb{R}$ interpolating the data $(x_{j,k}, f_{j,k})$ in the following manner

$$g_j \text{ is affine on } [x_{j,k}, x_{j,k+1}]\ , \qquad g_j(x_{j,k}) = f_{j,k} \tag{1.13}$$

Then the limit function g is the limit of g_j as j goes to infinity. To end this section we will briefly introduce the concept of basis functions $\phi_i(x)$, or basic limit functions. These are the limit functions of the scheme applied to cardinal data. Cardinal data is often expressed by the Kroenecker delta δ_i which denotes the vector that is one at index i and zero elsewhere.

$$\delta_i = [\ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0]$$

For stationary subdivision schemes all the basic limit functions are translates of one basic limit function $\phi(x)$.

$$\phi_i(x) = \phi(x - i) \tag{1.14}$$

For the stationary four-point scheme, this basic limit function is illustrated in Figure 1.3. Another convenient attribute of the basic limit function is that every limit function of a convergent subdivision scheme can be expressed as a linear combination of basic limit functions [19].

$$g(x) = \sum_i f_{0,i}\phi_i(x) \tag{1.15}$$

In this section we have introduced the three main assumptions on a subdivision scheme, which are independent of the positioning of the points. The schemes in question have been *local, bounded, affine.* We have also commented on the fact that the mask of a scheme may be dependent on the spatial position of the points, i.e the parametrisation, resulting in non-uniform schemes. Then we do not have one subdivision mask, but several masks. When we introduce the irregular four-point scheme, we will see that this is the case. We have provided a formulation of a general subdivision scheme and introduced the four-point scheme, which we will continue to use in order to demonstrate different properties later on.

1.2 Parametrisations

In the previous section we introduced properties of the subdivision scheme which where independent of the positioning of the original and refined values. We define the intermediate subdivision curves as the linear interpolant to the function values at level j. However, when defining the notion of convergence of these curves to a continuous function, we see that the interpolants depend on how we parametrise the refined function values. For example, if the subdivision rules at each level are dependent of the positioning of the values, and the values are not equally spaced, the subdivision scheme is no longer uniform, and extra caution must be taken when considering convergence. In a historical perspective the main focus have been on subdivision where the initial values are equally spaced and the the refinement in the parameters is dyadic, see e.g [1, 5, 9] resulting in stationary subdivision schemes. In later works the effects of more irregularity in the parametrisation and refinements have been further investigated [24, 4, 3] yielding non-stationary subdivision schemes. The set of parametrisations we consider is decribed through the very general notion of a multi-level grid inspired by [4].

Definition 1.3 (Multi-level grid). *Let $X_0 = \{x_i\}$ be a strictly increasing sequence of real numbers. Then the set of strictly increasing sequences in $\mathbb{R}$, $\mathbf{X} = \{X_j = \{x_{j,k}\}\ j \geq 0\}$ with the property that $X_j \subset X_{j+1}\ \forall\ j$ and*

$$\lim_{k\to-\infty} x_{0,k} = -\infty, \qquad \lim_{k\to\infty} x_{0,k} = \infty$$

is called a multi-level grid.

Definition 1.4 (Regular multi-level grid). *A multi-level grid $\mathbf{X}$ where*

$$X_0 = r\mathbb{Z},\ r\ \in\ \mathbb{R}_+$$

and

$$x_{j+1,2k+1} = \frac{1}{2}(x_{j,k+1} + x_{j,k})\ \forall\ j,k$$

is called regular.

Definition 1.5 (Semi-regular multi-level grid). *A multi-level grid $\mathbf{X}$ where X^0 is an arbitrary sequence while*

$$x_{j+1,2k+1} = \frac{1}{2}(x_{j,k+1} + x_{j,k})\ \forall\ j > 0,\ \forall\ k$$

is called semi-regular.

We will use the following definition $h_{j,k} = x_{j,k+1} - x_{j,k}$.

Definition 1.6 (Dyadically balanced multi-level grid [4]). *Given a multi-level grid $\mathbf{X}$, let*

$$\lambda = \sup_{j,k} \max\left(\frac{h_{j+1,2k}}{h_{j,k}}, \frac{h_{j+1,2k+1}}{h_{j,k}}\right)$$

If $\lambda < 1$ the multi-level grid is termed dyadically balanced

The regular and semi-regular multi-level grids are both dyadically balanced, since $\lambda = \frac{1}{2}$ in both cases. The definition below arises from [15], and we introduce it here, since we later will uncover the smoothness results related to these grids.

Definition 1.7 (Quasi-regular multi-level grid [15])**.** *A multi-level grid* $\mathbf{X}$ *is said to be quasi-regular if there exist positive numbers* a, b*, such that*

$$a2^{-j} \leq h_{j,k} \leq b2^{-j}, j \geq 0, k \in \mathbb{Z} \tag{1.16}$$

A regular multi-level grids is quasi-regular by $a = b = 1$. A semi-regular grid is quasi-regular, with a and b such that $a \leq h_{0,k} \leq b$.

Definition 1.8 (Homogeneous multi-level grid [4])**.** *Define the quantity* $\gamma, \gamma \geq 1$ *relating the neighbouring intervals by*

$$\gamma = \sup_{j,k} \frac{\max(h_{j,k+1}, h_{j,k-1})}{h_{j,k}}.$$

A multi-level grid $\mathbf{X}$ *is termed homogeneous if* $\gamma < \infty$*.*

A homogeneous multi-level grid is dyadically balanced, whereas a dyadically balanced multi-level grid need not be homogeneous. The example in [4] illustrates this well: If $x_0 = \mathbb{Z}$, while $x_{j+1,2k+1} = x_{j,k} + h_{j,k}/3$, then $h_{j,0} = \frac{1}{3^j}$ and $h_{j,-1} = \left(\frac{2}{3}\right)^j$, hence $\gamma = \infty$. Moreover, it follows that a quasi-regular grid is homogeneous with $\gamma \leq \frac{b}{a}$. In this thesis, as well as in this article [4], uniform bounds on the initial intervals are assumed as well, $\inf_k h_{0,k} > 0$ and $\sup_k h_{0,k} < \infty$. A multi-level grid is sometimes abbrivated to a grid.

1.3 Convergence

We have previously introduced the notion of a subdivision scheme, and can loosely state that the limit function is the limit of the linear interpolants as j goes to infinity. But when is the limit function continuous? We will try to answer this question in the following section. For the control polygon on level j, recall that we defined the affine function $g_j : \mathbb{R} \mapsto \mathbb{R}$ interpolating the data $(x_{j,k}, f_{j,k})$ in the following manner

$$g_j \text{ is affine on } [x_{j,k}, x_{j,k+1}] \,, \qquad g_j(x_{j,k}) = f_{j,k} \tag{1.17}$$

Then the limit function g is defined as

$$g(x) = \lim_{j \to \infty} g_j(x) \tag{1.18}$$

The initial values, $\mathbf{f}_0$, are bounded and assumed to be finitely supported . Hence they can be indexed $0, \ldots, n$ for the indices for which $f_i \neq 0$. This implies that all g_j are continuous and defined over a closed interval in $\mathbb{R}$, hence also bounded, and trivially compactly supported on $[x_0, x_n]$. We therefore consider the function space

$$C[x_0, x_n] = \{f : [x_0, x_n] \mapsto \mathbb{R} \mid f \text{ continuous}\}$$

equipped with the following metric

$$\|f - g\| = \sup_{x \in \mathbb{R}} |f(x) - g(x)|, \tag{1.19}$$

called the supremums metric. $C[x_0, x_n]$ is complete under this metric, for proof of this see e.g. [16]. We will need the following matrix norm

$$\|A\|_\infty = \sup_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} |A_{k,l}| \tag{1.20}$$

called the maximum absolute row sum norm. We are now ready to define the notion of a convergent scheme, and with it convergence with respect to a given multi-level grid.

Definition 1.9 (A convergent subdivision scheme [15])**.** *We say that* $\mathbf{S}$ *converges with respect to a multi-level grid* $\mathbf{X}$ *if, for any bounded* $\mathbf{f}_0$*, the corresponding sequences* g_j*, converges uniformly on* $\mathbb{R}$*. In other words, that for any* $\epsilon > 0$ *there exist a* $g \in C[x_0, x_n]$ *and* $J \in \mathbb{Z}$ *such that*

$$\|g_j - g\| < \epsilon \;\forall\; j \;\geq\; J \tag{1.21}$$

Moreover, we say that the scheme converges *if there exists a multi-level grid* $\mathbf{X}$ *for which* $\mathbf{S}$ *converges.*

We state the following auxilary lemma, sometimes useful when proving that the functions $\{g_j\}$ form a Cauchy sequence.

Lemma 1.1. *Given a sequence $\{s_j\}$ in $C[a,b]$. Suppose there are constants $C > 0$ and $\beta \in (0,1)$, such that*

$$\|s_{j+1} - s_j\| \leq C\beta^j, \qquad j \geq 0. \tag{1.22}$$

Then the sequence is Cauchy, so there exits a limit function $s \in C[a,b]$

$$s(x) := \lim_{j\to\infty} s_j(x) \tag{1.23}$$

We have the following sufficient condition for convergence.

Theorem 1.1 (A sufficient condition for convergence [15])**.** *Given* **S** *and assume that the associated difference scheme* **D** *satisfies the following property: there exist $J, K \geq 0$ and $\mu \in (0,1)$ such that*

$$\|D_{j+K}\ldots D_{j+1}D_j\|_\infty \leq \mu \ \forall \ j \ \geq \ J. \tag{1.24}$$

Then the scheme converges with respect to any grid.

To demonstrate the notion of convergence we consider a case of a stationary scheme over a regular grid.

The simplest case

A stationary scheme **S** is a uniform scheme associated with a regular multi-level grid, where $S_j = S, j \ \geq \ 0$, where Then the difference scheme **D** is also stationary $D = D_j, \ j \ \geq \ 0$. For a stationary scheme we have both necessary and sufficient conditions for convergence. Now we divide the mask into the even stencil $a = \{a_i\}_{i=0}^{a_S-1}$ and the odd stencil $b = \{b_k\}_{k=0}^{b_S-1}$ such that $b_S + a_S = 2n_S + 1$.

Proposition 1.1 (A necessary condition for convergence for a stationary subdivision scheme)**.** *Suppose the stationary scheme* **S** *converges for some non-trivial initial data* $\mathbf{f}_0$ *and the limit function $f \neq 0$. Then the mask satisfies*

$$\sum_{i=0}^{a_S-1} a_i = \sum_{k=0}^{b_S-1} b_k = 1 \tag{1.25}$$

Proof. Consider $f(x_0)$, for simplicity and without loss of generalization[4], and assume $f(x_0) \neq 0$, we have

$$f_{j+1,0} = \sum_i a_i f_{j,i} \tag{1.26}$$

$$f_{j+1,1} = \sum_k b_k f_{j,k} \tag{1.27}$$

[4]The schemes are uniform, so all points are refined by the same rules

Then by uniform convergence, both $f_{j+1,0}$ and $f_{j+1,1}$ converge to $f(x_0)$ as $j \to \infty$. Hence

$$0 = \lim_{j\to\infty} |f_{j+1,0} - f_{j+1,1}| = |\sum_{i=0}^{\max\{a_S,b_S\}} (a_i - b_i) f_{j,i}|,$$

which implies (1.25). □

This proposition tells us that if the stationary subdivision scheme converge then the new control points are formed by affine combinations of the old. For the rest of the section we will assume that the conditions for this proposition applies and recall the difference scheme defined in the previous section. We can now define the sufficient condition for convergence of a stationary subdivision scheme.

Theorem 1.2 (A sufficient condition for convergence, [15]). *Assume that the stationary subdivision scheme* **S** *reproduces constants and its difference scheme* **D** *satisfies the following property; there exist $K \geq 0$ and $\mu \in (0,1)$ such that*

$$\|D^K\|_\infty \leq \mu \tag{1.28}$$

Then the scheme converges.

Proof. Recall the definition of the supremum norm above, and consider the affine function g_j defined above. Fix $j \geq K$. We will show that the sequence $\{g_j\}$ form a Cauchy sequence. All sums are over a finite subset of $\mathbb{Z}$. The following reformulation will be used several times, and follows from Proposition (1.1); assume $\sum_k c_k = 1$, then obviously

$$f_{j,i} = f_{j,i} \sum_k c_k = \sum_k c_k f_{j,i}$$

Let us consider the maximum difference between g_{j+1} and g_j. Since they are both affine the difference is given as

$$\|g_{j+1} - g_j\|_\infty = \max_i \{|f_{j+1,2i} - f_{j,i}|, |f_{j+1,2i+1} - \frac{1}{2}(f_{j,i} + f_{j,i+1})|\} \tag{1.29}$$

We consider the terms separately

$$\begin{aligned} f_{j+1,2i+1} - \frac{1}{2}(f_{j,i} + f_{j,i+1}) &= \sum_k b_k (f_{j,i+k} - \frac{1}{2}(f_{j,i} + f_{j,i+1})) \\ &= \sum_k \hat{b}_k \Delta f_{j,i+k} \end{aligned}$$

and likewise

$$\begin{aligned} f_{j+1,2i} - f_{j,i} &= \sum_k a_k (f_{j,i+k} - f_{i,k}) \\ &= \sum_j \hat{a}_j \Delta f_{i+j} \end{aligned}$$

combined this gives

$$\|\mathbf{f}_{j+1} - \mathbf{f}_j\|_\infty \le C_1 \max_i |\Delta f_{j,i}|$$

where $C_1 = \max\{\sum_i |\hat{b}_i|, \sum_i |\hat{a}_i|\}$. Then by our assumption and that $\Delta\mathbf{f}_{j+1} = D_j \Delta\mathbf{f}_j$,

$$\max_i |\Delta f_{j,i}| \le \mu^{j/K} \max_i |\Delta f_{j-K,i}|$$

Hence,

$$\max_i |\Delta\mathbf{f}_{j,i}| \le C_2 \mu^{j/K}$$

where $C_2 = \max_i \Delta f_{j-K,i}$. Therefore

$$\|g_{j+1} - g_j\|_\infty \le C\mu^{j/K} \tag{1.30}$$

with $C = C_1 C_2$ implies that the sequence is Cauchy, according to Lemma 1.1. So the scheme converges. □

To summarize: the necessary condition states that a convergent scheme has stencils which sum to one, ensuring affine invariance. The sufficient conditions tells us that if we can uniformly bound the first order differences of the control points then the scheme is convergent. Now we have discussed the simplest case; where the scheme is stationary. The sufficient condition gives us a general algorithm of finding out if a scheme is convergent.

Example 1.3. *Back to our stationary four-point scheme. Then the matrix of the difference scheme takes the following form*

$$D = \begin{pmatrix} \ddots & & & & & & & \\ \cdots & -\frac{1}{16} & \frac{1}{2} & \frac{1}{16} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \frac{1}{16} & \frac{1}{2} & -\frac{1}{16} & 0 & 0 & \cdots \\ \cdots & 0 & -\frac{1}{16} & \frac{1}{2} & \frac{1}{16} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{1}{16} & \frac{1}{2} & -\frac{1}{16} & 0 & \cdots \\ \cdots & 0 & 0 & -\frac{1}{16} & \frac{1}{2} & \frac{1}{16} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{1}{16} & \frac{1}{2} & -\frac{1}{16} & \cdots \\ & & & & & & & \ddots \end{pmatrix} \tag{1.31}$$

We immediately see that $\|D\|_\infty = \frac{5}{8} < 1$ hence the scheme is convergent with $K = 1$ according to Theorem 1.2.

General convergence

It is natural to impose Proposition 1.1 on schemes defined over a irregular multi-level grids as well. This ensures that the limit function is independent of the coordinate system. Schemes that do not produce such limit function are of little practical relevance [3]. In [4] the convergence of a class of irregular interpolatory subdivision schemes, which generalise the Dubuc-Deslauriers schemes, is found by considering derived schemes, which will be defined shortly. But the main conclusion for us is that the four-point scheme converge for both dyadically balanced and homogeneous multi-level grids. For a general difference scheme to satisfy (1.24) [15] suggests a method using the notions of equivalent schemes.

Definition 1.10 (Equivalent subdivision schemes[15])**.** *We say that two subdivision schemes* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *are* equivalent *if there exist two positive numbers* α, β *such that*

$$\|S_j - \tilde{S}_j\| \leq \alpha 2^{-\beta j}, \qquad j \geq 0. \tag{1.32}$$

Using this definition it is established in [15] that two subdivision schemes $\mathbf{S}$, $\tilde{\mathbf{S}}$ that are equivalent, if one satisfies the sufficient condition for convergence (1.24), then the other will as well. In [15] it is proven that the four-point scheme over a quasi-regular grid it equivalent to the four-point scheme over a regular grid, so the four-point scheme over a quasi-regular grid converges. Recall that we found the regular four-point scheme to converge satisfying (1.28) with $\tilde{K} = 1$ and $\tilde{\nu} = \frac{5}{8}$ in Example 1.3. A function is said to be Hölder-continuous with exponent $\nu \in (0, 1)$ if there exists $M > 0$ s.t

$$|f(y) - f(x)| \leq M|y - x|^{\nu}, \qquad x, y \in \mathbb{R} \tag{1.33}$$

It is shown in [15] that the limit functions for the four-point scheme over a quasi-regular multi-level grid are Hölder-continuous with exponent ν for any $\nu < \hat{\nu}$, where $\hat{\nu} = -\frac{1}{K+1} \log_2(\tilde{\nu}) \approx 0.334$.

1.4 Smoothness

Now that we have introduced sufficient conditions for a subdivision scheme to produce a continuous limit function, the question of smoothness remains. As questioned in [19]: What is the highest continuous derivative of the limit function, or the lowest discontinuous derivative? More precisely, what are the bounds of the Hölder exponent? An n times differentiable and bounded function f is termed Hölder-continuous with exponent $n + \alpha$, with $\alpha \in (0, 1)$ if there exist a constant $C > 0$ such that

$$\frac{|f^{(n)}(y) - f^{(n)}(x)|}{|y - x|^{\alpha}} \leq C \qquad \forall \ \leq \ x, y \ \in \ \mathbb{R}, \tag{1.34}$$

The notations, which we will use several times, $f \in C^{1+1}$ or $f \in C^{2-\epsilon}$ implies that $f'(x)$ is Hölder-continuous with exponent $1 - \epsilon$ for any $\epsilon > 0$. Over the years, various techniques have been developed in order to answer the question of smoothness for different subdivision schemes. The different techniques are based on spectral analysis [24, 19], the symbol[5] of the scheme [8] and the reduction strategy [4, 3] and with it the notion of derived schemes. In short, spectral analysis deals with the smoothness by considering the spectral properties of a finite dimensional portion of the infinite subdivision matrix. By different conditions on the eigenpairs, in terms of dominance and multiplicity of eigenvalues we can deduce smoothness results for a stationary subdivision scheme over a regular grid. In [24] it was shown how to use smoothness results in the regular case to estimate the smoothness over a semi-regular multi-level grid, and we revisit his approach in the last section of this chapter. The technique involving the symbol uses the mask of a regular scheme to deduce the smoothness of the subdivision scheme. We will not consider this approach, but an interested reader may consult e.g [8, 19]. While spectral analysis and analysis of algebraic properties of the symbol is quite powerful and we get good, in some cases optimal, bounds on the Hölder exponent, they can not be adapted directly to subdivision schemes over irregular multi-level grids [4]. The reduction strategy, on the other hand, we consider as the most general strategy and can be applied to the schemes under our assumptions, that is being *local, bounded and affine* [4, 3] and possibly non-uniform and non-stationary. Derived schemes are subdivision schemes relating the divided differences across refinement levels. There schemes are the essential ingredients in the reduction strategy and in contrast to difference schemes, derived schemes are dependent on the parametrisation, which we will see shortly. But let us first give two examples of subdivision schemes and the smoothness results we have for these schemes.

[5] the symbol is the z-transform of the mask

Examples

The Dubuc-Deslauriers schemes is a family of interpolatory subdivision schemes presented in [6, 5] and are defined of any $d \in \mathbb{Z}$, by the following construction

$$\begin{aligned} f_{j+1,k} &= f_{j,k}, \\ f_{j+1,2k+1} &= p_{j,k}^{[2d+1]}(x_{j+1,2k+1}), \end{aligned} \tag{1.35}$$

where $p_{j,k}^{[2d+1]}(x)$ is the unique polynomial interpolant of degree $2d+1$ to $g_{j,k-d}, \ldots, g_{j,k+d+1}$. Furthermore $2d+1$ is termed the degree of the scheme. In addition it is easy to verify that the scheme reproduce all polynomials of degree $\leq 2d+1$. Another famous example of an interpolatory subdivision scheme is the four-point scheme of [9] defined as

$$\begin{aligned} f_{j+1,2k} &= f_{j,k}, \\ f_{j+1,2k+1} &= -\omega(f_{j,k-1} + f_{j,k+2}) + (\omega + \frac{1}{2})(f_{j,k} + f_{j,k+1}) \end{aligned} \tag{1.36}$$

where ω is conventionally termed the tension parameter. For $\omega = \frac{1}{16}$ the four-point scheme coincides with the Dubuc-Deslauriers scheme of order 3, and hence reproduce cubic polynomials. It is shown that the limit curve of the four-point scheme is C^{1+1} for $0 < \omega < \frac{1+\sqrt{5}}{8}$ [7]. In [13] it is shown by the means of spectral analysis that the limit function is C^1 if and only if $0 < \omega < \omega^*$, where $\omega^* \approx 0.19278$. The four-point scheme with tension parameter $\omega = \frac{1}{16}$ is formulated as

$$\begin{aligned} f_{j+1,2k} &= f_{j,k} \\ f_{j+1,2k+1} &= -\frac{1}{16}(f_{j,k-1} + f_{j,k+2}) + \frac{9}{16}(f_{j,k} + f_{j,k+1}), \end{aligned} \tag{1.37}$$

which was the scheme introduced in the first section. Whenever we refer to the stationary four-point scheme assume that it is given with tension parameter $\omega = \frac{1}{16}$, as in (1.37). The general form of the four-point scheme allowing irregular parametrization takes the form of (1.35) of degree 3. As we can see from the definitions, calculating the first odd point in the first iteration $x_{1,1}$ requires that the quantities $f_{0,-2}, f_{0,-1}$ are known, and likewise, calculation of the last odd point $x_{1,n-1}$ requires the quantities $f_{0,n+1}, f_{0,n+2}$. This implies that in order to define the curve on the entire interval $[x_{0,0}, x_{0,n}]$ we have to provide four extra conditions which will influence the resulting curve.

A well-known example of approximative schemes is the family of B-spline schemes of degree d over regular grids. These are schemes known to generate the spline curve of degree d and has the known smoothness of C^{d-1} at the knots and C^∞ elsewhere, since splines are piecewise polynomial. For of a general degree d the mask of the scheme is given as

$$m_j^d = \frac{1}{2^d}\binom{d+1}{j} \tag{1.38}$$

1.4.1 Smoothness of the irregular four-point scheme

In the following section we will present the smoothness results for the four-point subdivision scheme over the irregular grids introduced in section 1.2, based on the approaches and notations in [4] and [15]. In the previous sections, we have not restricted the reviews and results to only interpolatory subdivision, but we will do so henceforth. This is due to the fact that we only consider interpolatory subdivision in the rest of the thesis. The following section involves tedious mathematics and can be skipped on first read-through. Up to now have we viewed the limit function g as the limit of the piecewise linear interpolant to the intermediate control points, termed g_j. If we differentiate each g_j we are left with a sequence of piecewise constant functions, and differentiating twice gives a sequence of Dirac delta functions [19]. From these reflections it is not clear how we can define or study higher order smoothness from differentiation of the piecewise linear interpolants. In the paper by Daubechies, Guskov and Sweldens [4], the idea is to define the limit function as

$$g(x) = \lim_{j\to\infty} f_{j,k_j(x)} \tag{1.39}$$

where $k_j(x) = \max l : x_{j,k} \leq x$, the grid point closest to x from the left, and we will apply this approach in the following section. In order to find the smoothness of $g(x)$ we must study the functions

$$g^{[p]}(x) = \lim_{j\to\infty} f^{[p]}_{j,k_j(x)} \tag{1.40}$$

where $\mathbf{f}_j^{[p]}$ are the divided differences of order p, where $p \geq 0$ based on the sequence $\mathbf{f}_j$, and prove that these functions are indeed the derivatives of $g(x)$. To do so we introduce the notion of derived schemes, relating divided differences across refinement levels. The second to last derived scheme which converges, gives a lower bound on the Hölder exponent of the original scheme[19]. The rate of divergence of the last scheme can also provide us with an upper bound on the Hölder exponent, as we will see in this section. Fix a multi-level grid $\mathbf{X}$. We first consider the first order divided difference. From the definition of first order divided differences of $\mathbf{f}_j$ we have

$$f^{[1]}_{j,k} = [x_{j,k}, x_{j,k+1}]f_j \tag{1.41}$$

$$= \frac{f_{j,k+1} - f_{j,k}}{x_{j,k+1} - x_{j,k}} \tag{1.42}$$

Then the derived scheme associated with $\mathbf{S}$, where $\mathbf{S}$ is bounded, affine and local, $\mathbf{S}^{[1]} = \{S_j^{[1]}\}_{j\geq 0}$ is the scheme satisfying

$$\mathbf{f}^{[1]}_{j+1} = S_j^{[1]}\mathbf{f}_j^{[1]}, \qquad j \geq 0 \tag{1.43}$$

where

$$S^{[1]}_{j,l,k} = \frac{h_{j,k}}{h_{j+1,l}} D_{j,l,k}. \tag{1.44}$$

The dependence on $\mathbf{X}$ is now evident and $\mathbf{S}^{[1]}$ inherits locality from $\mathbf{D}$. For the construction of $S_j^{[1]}$ to be valid, we must demand that $\mathbf{S}$ is affine. We can also define the difference operator of order 1 relating the control values at level j with its divided difference.

$$\mathbf{f}_j^{[1]} = D_j^{[1]}\mathbf{f}_j \tag{1.45}$$

where

$$D_{j,l,k}^{[1]} = \begin{cases} -\frac{1}{h_{j,l}}, & \text{if } k = l, \\ \frac{1}{h_{j,l}}, & \text{if } k = l+1, \\ 0 & \text{oth.} \end{cases} \tag{1.46}$$

Now we have the following set of equations

$$\begin{aligned} \mathbf{f}_j^{[1]} &= D_j^{[1]}\mathbf{f}_j, \\ \mathbf{f}_{j+1} &= S_j\mathbf{f}_j, \\ \mathbf{f}_{j+1}^{[1]} &= S_j^{[1]}\mathbf{f}_j^{[1]}, \qquad j \geq 0 \end{aligned}$$

From the set of equations above we can state a second *commutation formula.*

$$S_j^{[1]}D_j^{[1]} = D_{j+1}^{[1]}S_j \tag{1.47}$$

We can continue in this manner for the higher order divided differences, and find subdivision schemes relating divided differences across refinement levels. We have the following notion of order of the scheme, taken from the article by Maxim and Mazure [15]: We say that $\mathbf{S}$ is *of order greater than or equal to* p if we have been able to define a local and bounded subdivision scheme $\mathbf{S}^{[p-1]}$ enabling us to calculate the divided differences of order $(p-1)$ of $\mathbf{f}_j$. Further we say that $\mathbf{S}^{[p-1]}$ is of order greater than or equal to 1 if it reproduces constants. As we consider the irregular four-point sheme, it is important to point out that for the Dubuc-Deslauriers schemes of degree n the order of the scheme is also n since it is known to reproduce polynomials up to degree n [4]. Moreover, generally, the order of a subdivision scheme cannot exceed $2n_S$, since the masks of the schemes decrease in size by one for each iteration. Above we denoted the limit function of these derived schemes $\mathbf{S}^{[p]}$ as

$$g^{[p]}(x) = \lim_{j\to\infty} g_{j,k_j(x)}^{[p]} \tag{1.48}$$

where $k_j(x) = \max\{l : x_{j,k} \leq x\}$. We define the higher order divided differences as in [4];

$$g_{j,k}^{[p]} = \frac{g_{j,k+1}^{[p-1]} - g_{j,k}^{[p-1]}}{h_{j,k}^{[p]}} \tag{1.49}$$

$$[x_{j,k}, x_{j,k+1}, \dots, x_{j,k+p}]g_j = \frac{[x_{j,k+1}, x_{j,k+2}, \dots, x_{j,k+p}]g_j - [x_{j,k}, x_{j,k+1}, \dots, x_{j,k+p-1}]g_j}{x_{j,k+p} - x_{j,k}} \tag{1.50}$$

We can continue like we did for the first order difference, in an inductive manner, and define the derived schemes componentwise as

$$S^{[p]}_{j,k,m} = \frac{h^{[p]}_{j,m}}{h^{[p]}_{j+1,k}} \sum_{l>m} (S^{[p-1]}_{j,k+1,l} - S^{[p-1]}_{j,k,l})$$

as long as $\mathbf{S}^{[p-1]}$ reproduce constants. Using this for two consequtive elements gives

$$\frac{S^{[p]}_{j,l,m-1}}{h^{[p]}_{j,m-1}} - \frac{S^{[p]}_{j,l,m}}{h^{[p]}_{j,m}} = \frac{S^{[p]}_{j,l+1,m} - S^{[p]}_{j,l,m}}{h^{[p]}_{j+1,l}}$$

It can be shown that the matrices of the derived schemes satisfy [4]:

$$S^{[p]}_j D^{[p]}_j = D^{[p]}_{j+1} S^{[p-1]}_j$$

and

$$S^{[p]}_j \Delta^{[p]}_j = \Delta^{[p]}_{j+1} S_j$$

where $D^{[p]}_j$ are the difference schemes of order p. The overall goal is to find an expression for the highest order scheme relating the divided differences across refinement levels to investigate the behavior of the sequences at j goes to infinity.

$$f^{[p]}_{j+1} = S^{[p]}_j f^{[p]}_j$$

Once we have found this, we seek an estimate on the bound of the rate of growth or decay of the divided difference we consider. Then we transform this estimate to a bound on the lower-order divided differences through a reduction strategy described in detail in [4] and briefly below. We illustrate the reduction strategy through an example, again based on the four-point scheme, as general results are quite involved and considered beyond the scope of this thesis. The odd stencils of the irregular four-point scheme are given as

$$S_{j,2k+1,k+u} = \prod_{-2<v<2, v\neq u} \frac{x_{j+1,2k+1} - x_{j,k+v}}{x_{j,k+u} - x_{j,k+v}} \tag{1.51}$$

We see now that the scheme is no longer uniform, the stencils are dependent of the location of the new odd point $x_{j+1,2k+1}$. We will prove the following result given in [4] regarding regularity estimates in the case of the four-point scheme over a homogeneous multi-level grid using the reduction strategy. Keep in mind that a homogeneous grid is dyadically balanced, and that our definition differs from [4] where they use β as the quantity, here $\lambda = 1 - \beta$.

Theorem 1.3 (Regularity of the four-point scheme over a homogeneous multi-level grid)**.** *Consider the four-point scheme over a homogeneous multi-level grid. The limit* *functions* $g^{[p]}$ *as defined in (1.48) are well-defined and continuous for all* $p = \{0, 1\}$. *Moreover* $g^{[1]}$ *is Hölder-continuous with exponent* $\alpha = \frac{\log(\lambda)}{\log(1-\lambda)} - \epsilon$ *for any* $\epsilon > 0$. *The rate of convergence is exponential:*

$$|g^{[1]}(y) - f^{[1]}_{j,k_j(y)}| \leq C'\lambda^j \tag{1.52}$$

And finally, one has

$$g^{[1]}(y) = \frac{dg(y)}{dy} \tag{1.53}$$

To prove this theorem we need the following three lemmas which are part of the machinery behind the reduction strategy in the homogeneous case.

Lemma 1.2 (Lemma 1[4]). *For* $\alpha > 0, \sigma \geq 0, r \in \mathbb{R}$, *the bound*

$$|f^{[p]}_{j,k}| \leq Cj^{\sigma}\frac{\alpha^j}{(h_{j,k})^r} \tag{1.54}$$

is equivalent to the bound

$$|\tilde{f}^{[p]}_{j,k}| \leq C'j^{\sigma}\frac{\alpha^j}{(h_{j,k})^{r-1}} \tag{1.55}$$

Lemma 1.3 (Lemma 2[4]). *Suppose that, for some* $\alpha > 0, \sigma \geq 0, r \in \mathbb{R}$,

$$|\tilde{f}^{[p+1]}_{j,k}| \leq Cj^{\sigma}\frac{\alpha^j}{(h_{j,k})^{r-1}} \tag{1.56}$$

also let the coefficients of the subdivision scheme $\mathbf{S}^{[p]}$ *be bounded uniformly in* j, k, l. *Then*

$$|f^{[p]}_{j+1,2k+s} - f^{[p]}_{j,k}| \leq Cj^{\sigma}\frac{\alpha^j}{(h_{j,k})^r} \tag{1.57}$$

for $s \in \{0, 1\}$.

Lemma 1.4 (Lemma 3[4]). *Suppose that, for some* $\alpha > 0, \sigma \geq 0, r \geq 0$,

$$|f^{[p]}_{j+1,2k+s} - f^{[p]}_{j,k}| \leq Cj^{\sigma}\frac{\alpha^j}{(h_{j,k})^r}$$

for $s \in \{0, 1\}$. *Then*

$$|f^{[p]}_{j,k}| \leq C'\left[j^{\sigma+\omega}\frac{\hat{\alpha}^j}{(h_{j,k})^r} + 1\right], \tag{1.58}$$

where $\hat{\alpha} = \max\{\alpha, \lambda^j\}$ *and,*

$$\omega = \begin{cases} 0 \text{ if } \alpha \neq \lambda^r \\ 1 \text{ if } \alpha = \lambda^r \end{cases} \tag{1.59}$$

The proofs of the lemmas above are given in the cited article, but we will use them to prove the Theorem 1.3. The following proof is based on the approach in [4].

Proof of Theorem 1.3. We find the highest order derived scheme, $S^{[4]}$, as the order of the four-point scheme is 3. As mentioned in [4], the expressions for $S^{[4]}$ are quite involved for this scheme, so we will consider the scheme for the differences of the divided differences instead. Let us introduce a short hand for differences of divided differences.

$$\tilde{g}^{[p]}_{j,k} = g^{[p-1]}_{j,k+1} - g^{[p-1]}_{j,k} = h^{[p]}_{j,k} g^{[p]}_{j,k}$$

Hence we we find the scheme $\mathbf{T}$ relating $\tilde{f}^{[4]}_{j+1}$ and $\tilde{f}^{[4]}_{j}$. The matrices of $\mathbf{T}$ have the following entry in the even rows

$$T_{j,2k-2,k-2} = \frac{h^{[4]}_{j+1,2k-2}}{h^{[2]}_{j+1,2k-1}} \tag{1.60}$$

while the odd rows have

$$T_{j,2k-1,k-2} = -\frac{h^{[1]}_{j+1,2k-2}}{h^{[2]}_{j+1,2k-1}} \tag{1.61}$$

$$T_{j,2k-1,k-1} = -\frac{h^{[1]}_{j+1,2k+3}}{h^{[2]}_{j+1,2k+1}} \tag{1.62}$$

Assume $s \in \{0,1\}$. For a homogeneous scheme, all the coefficients of $\mathbf{T}$ are uniformly bounded in j, and give us a bound on $\tilde{f}^{[4]}_{j,k}$,

$$|\tilde{f}^{[4]}_{j,k}| \leq C\frac{\lambda^j}{(h_{j,k})^2}$$

which by Lemma 1.3 can be lifted into a bound on the third order divided differences across refinement levels.

$$|f^{[3]}_{j+1,2k+s} - f^{[3]}_{j,k}| \leq C\frac{\lambda^j}{(h_{j,k})^2}. \tag{1.63}$$

By Lemma 1.4 we have

$$|f^{[3]}_{j,k}| \leq C\frac{\lambda^j}{(h_{j,k})^2} + C \tag{1.64}$$

where the constant C can be neglected as the first term will dominate as $j \to \infty$ [4].

$$|f^{[3]}_{j,k}| \leq C\frac{\lambda^j}{(h_{j,k})^2} \tag{1.65}$$

Now we have done one iteration in the reduction strategy, and can apply Lemma 1.2 to reduce the order of divided differences further. (1.65) is equivalent to

$$|\tilde{f}^{[3]}_{j,k}| \leq C\frac{\lambda^j}{(h_{j,k})} \tag{1.66}$$

Then by Lemma 1.3 again, we have

$$|f^{[2]}_{j+1,2k+s} - f^{[2]}_{j,k}| \leq C'\frac{\lambda^j}{(h_{j,k})} \tag{1.67}$$

By an application of Lemma 1.4 we arrive at a bound for the second order divided difference

$$|f^{[2]}_{j,k}| \leq Cj\frac{\lambda^j}{h_{j,k}} \tag{1.68}$$

By applying Lemmas 1.2–1.3 once more we are left with a bound on the first order divided differences.

$$|f^{[1]}_{j+1,2k+s} - f^{[1]}_{j,k}| \leq Cj\lambda^j \tag{1.69}$$

Intuitively we see that $j\lambda^j \to 0$ as $j \to \infty$ since $\frac{1}{2} \leq \lambda < 1$ and λ^j goes much faster to zero that j goes to infinity, but for completeness we will provide a proof of this statement. Consider the real valued function l

$$l(x) = \frac{x}{\frac{1}{\lambda^x}} \tag{1.70}$$

and assume $|\lambda| < 1$. $l(x)$ is well-defined for all $x \in \mathbb{R}$.

$$\lim_{x\to\infty} l(x) = \lim_{x\to\infty} \frac{x}{\frac{1}{\lambda^x}} \tag{1.71}$$

is a $\frac{\infty}{\infty}$ expression, so we can use L'Hopital's rule. Let $l_1(x) = x$ and $l_2(x) = \lambda^{-x}$. Then by differentiation we have

$$\lim_{x\to\infty} \frac{l_1'(x)}{l_2'(x)} = \lim_{x\to\infty} \frac{1}{\frac{1}{-\ln(\lambda)\lambda^x}} \tag{1.72}$$

$$= \lim_{x\to\infty} -a\lambda^x = 0. \tag{1.73}$$

Then we can conclude that $\lim_{j\to\infty} l(x)|_{\mathbb{Z}} = 0$ and state that $\mathbf{f}^{[1]}_j$ form a Cauchy sequence by (1.77) and hence converge. Recall that we denoted the limit function of a derived subdivision scheme $\mathbf{S}^{[1]}$ as the pointwise limit of the divided difference of order 1.

$$g^{[1]}(x) = \lim_{j\to\infty} f^{[1]}_{j,k_j(x)}, \tag{1.74}$$

where $k_j(x) = \max\{l : x_{j,l} \leq x\}$. Convergence of $\mathbf{f}_j^{[1]}$ shown in 1.77, proves that this function is well-defined. But according to the theorem we must also show that this function is continuous. Fix $y, y+t \in \mathbb{R}$ such that $|t| < \min_k h_{0,k} = h_\star$. We must prove that we can get $|g^{[1]}(y+t) - g^{[1]}(x)|$ arbitrary small. Define $h_j(y) = \min\{h_{j,k_j(y)-1}, h_{j,k_j(y)}, h_{j,k_j(y)+1}\}$, then there exists an index j such that

$$h_{j+1}(y) \leq |t| < h_j(y) \tag{1.75}$$

then it follows that

$$|k_j(y+t) - k_j(y)| \leq 1 \tag{1.76}$$

So we can conclude by Lemma 1.2 what

$$|f^{[1]}_{j,k_j(y+t)} - f^{[1]}_{j,k_j(y)}| \leq Cj\lambda^j \tag{1.77}$$

By homogeneity we have

$$C_1|t| \leq h_{j,k_j(y)+s} \leq C_2|t| \tag{1.78}$$

for $s \in \{-1, 0, 1\}$. Moreover, for any $z \in \mathbb{R}$ we have

$$|g^{[1]}(z) - f^{[1]}_{j,k_j(z)}| \leq C\lambda^j \tag{1.79}$$

Showing the exponential rate of convergence to the limit function. And by the results above and the triangle inequality we have stated that

$$\begin{aligned} &|g^{[1]}(y+t) - g^{[1]}(x)| \leq \\ |g^{[1]}(y+t) - f^{[1]}_{j,k_j(y+t)}| + |f^{[1]}_{j,k_j(y+t)} &- f^{[1]}_{j,k_j(y)}| + |f^{[1]}_{j,kj_(y)} - g^{[1]}(x)| \\ &\leq (C + C'j)\lambda^j, \end{aligned}$$

which goes to zero as j goes to infinity. But from this inequality we can also find an expression for the Hölder exponent. Since the grid is homogeneous, it is dyadically balanced and we have the following inequalities

$$|t| \geq h_{j,k_j(y)} \geq (1-\lambda)^j h_\star \tag{1.80}$$

Hence we have that

$$\frac{|g^{[1]}(y+t) - g^{[1]}(x)|}{|t|^\alpha} \leq (C'j + C)\left(\frac{\lambda}{(1-\lambda)^\alpha}\right)^j, \tag{1.81}$$

which is bounded at a function of j if

$$\left(\frac{\lambda}{1-\lambda}\right) < 1 \Longleftrightarrow \alpha < \frac{\log(\lambda)}{\log(1-\lambda)}. \tag{1.82}$$

Then we can conclude that $g^{[1]}(y)$ is Hölder-continuous with any exponent $\alpha < \frac{\log(\lambda)}{\log(1-\lambda)}$. It remains to show that g is the limit of the sequence of function values $\mathbf{f}_j$, and we can employ the same procedure as above to do this. Finally it we must show that $g^{[1]}$ is the derivative of g. The proof of this is given in the cited article. $\square$

We have tried to illustrated the reduction strategy by means of an example. Moreover, for this thesis the relevant conclusions in [4] is that for a dyadically balanced grid with $\frac{1}{2} \leq \lambda \leq \frac{2}{3}$, the four-point scheme has a regularity of C^{1+1} in our notation above. In [11], the bound was improved; $g \in C^{1+1}$ when $\frac{1}{2} \leq \lambda \leq \lambda_0 \approx 0.7142$. An advantage of using the approach of Floater in [11], is that we find an expression for $\tilde{g}^{[4]}_{j,k}$, without having to find $g^{[i]}_{j,k}, i = 1, 2, 3$ first, as described above. For a merely dyadically balanced grid, the analysis becomes more delicate as illustrated in [4]. For a homogeneous multi-level grid with $\gamma \leq \gamma_0 \approx 2.49992$ the scheme was also proven to be C^{1+1}

In section 1.3 we introduced the notion of equivalent schemes, and discussed the results from [15], relating the convergence of the regular four-point scheme to convergence of the quasi-regular four-point scheme. In order to recover the differentiability results for the quasi-regular case the notion of equivalent schemes is revisited and a comparison of schemes is used to establish bound on the Hölder exponent. However, since the derived schemes are dependent of the grids, they also introduced the notion of equivalent grids.

Definition 1.11 (Equivalent multi-level grids [15]). *We say that two grids,* $\mathbf{X}$, $\tilde{\mathbf{X}}$, *are* equivalent *if, for any given positive integer* N, *there exist two positive numbers* γ, η, *such that for any* $k, l \in \mathbb{Z}$ *and any* $p \geq 1$

$$-N \leq 2l - k \leq N - p \Rightarrow \left| \frac{h^{[p]}_{j,l}}{h^{[p]}_{j+1,k}} - \frac{\tilde{h}^{[p]}_{j,l}}{\tilde{h}^{[p]}_{j+1,k}} \right| \leq \gamma 2^{-\eta j}, \qquad j \geq 0. \tag{1.83}$$

Let $\mathbf{S}$ denote the four-point scheme over a quasi-regular multi-level grid $\mathbf{X}$, and let $\tilde{\mathbf{S}}$ denote the four-point scheme over a regular multi-level grid $\tilde{\mathbf{X}}$. Then it can be proven that $\tilde{\mathbf{S}}$ and $\mathbf{S}$ are equivalent with respect to Definition 1.10, and that $\tilde{\mathbf{X}}$ and $\mathbf{X}$ are equivalent by Definition 1.11. We view the assumptions of Corollary 5.7 of [15], in order to prove the regularity of the four-point scheme over a quasi-regular grid, so we need to verify that each of the differences schemes of the derived subdivision schemes $\tilde{D}^{[p]}$ satisfies (1.28) for some $\tilde{K}_P$, and some $\tilde{\mu}_P$, $1 \leq p \leq 2$. Then we can conclude that limit function produced by $\mathbf{S}$ over $\mathbf{X}$ are Hölder-continuous with exponent $1 + \mu$, for any $\mu \in (0, -\frac{1}{\tilde{K}_2+1} \log_2(\tilde{\mu}_2))$. We found the difference scheme $\tilde{D}$ of $\tilde{S}$ to satisfy (1.28) with $\tilde{K}_1 = 1$ and $\tilde{\mu} = \frac{5}{8}$ in Example 1.3. Let us now look at $\tilde{D}^{[2]}$, the difference scheme for the first order divided differences of g_j. Locally the the scheme looks like

$$\Delta g^{[1]}_{j+1} = \tilde{D}^{[2]} \Delta g^{[1]}_j$$

$$\begin{pmatrix} \Delta g^{[1]}_{j+1,2k-1} \\ \Delta g^{[1]}_{j+1,2k} \\ \Delta g^{[1]}_{j+1,2k+1} \\ \Delta g^{[1]}_{j+1,2k+2} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -1 & 6 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 6 & -1 \\ 0 & 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} \Delta g^{[1]}_{j,k-2} \\ \Delta g^{[1]}_{j,k-1} \\ \Delta g^{[1]}_{j,k} \\ \Delta g^{[1]}_{j,k+1} \end{pmatrix}$$

Here we see that $\|\tilde{D}^{[2]}\| = 1$, so we cannot conclude that $\tilde{D}^{[2]}$ satisfies (1.28) for $\tilde{K}_2 = 1$. Lets consider $\tilde{K}_2 = 2$. Again we can locally construct the dependecies from level $j + 1$

to level j.

$$\Delta g^{[1]}_{j+2} = \tilde{D}^{[2]} \Delta g^{[1]}_{j+1}$$

$$\begin{pmatrix} \Delta g^{[1]}_{j+2,4k} \\ \Delta g^{[1]}_{j+2,4k+1} \\ \Delta g^{[1]}_{j+2,4k+2} \\ \Delta g^{[1]}_{j+2,4k+3} \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2 & 2 & 0 & 0 \\ -1 & 6 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 6 & -1 \end{pmatrix} \begin{pmatrix} \Delta g^{[1]}_{j+1,2k-1} \\ \Delta g^{[1]}_{j+1,2k} \\ \Delta g^{[1]}_{j+1,2k+1} \\ \Delta g^{[1]}_{j,2k+2} \end{pmatrix}$$

Combining the two sets of equations above we find the following local relation

$$\Delta g^{[1]}_{j+2} = (\tilde{D}^{[2]})^2 \Delta g^{[1]}_{j}$$

$$\begin{pmatrix} \Delta g^{[1]}_{j+2,4k} \\ \Delta g^{[1]}_{j+2,4k+1} \\ \Delta g^{[1]}_{j+2,4k+2} \\ \Delta g^{[1]}_{j+2,4k+3} \end{pmatrix} = \frac{1}{64} \begin{pmatrix} -2 & 16 & 2 & 0 \\ 1 & 7 & 7 & 1 \\ 0 & 2 & 16 & 2 \\ 0 & -8 & 32 & -8 \end{pmatrix} \begin{pmatrix} \Delta g^{[1]}_{j,k-2} \\ \Delta g^{[1]}_{j,k-1} \\ \Delta g^{[1]}_{j,k} \\ \Delta g^{[1]}_{j,k+1} \end{pmatrix}$$

Now we see that $\|(\tilde{D}^{[2]})^2\| = \frac{3}{4} < 1$, so $\tilde{D}^{[2]}$ satisfies (1.28) with $\tilde{K_2} = 2$, $\tilde{\mu}_2 = \frac{3}{4}$. Then by Corollary 5.7 of [15] we can conclude that the four-point scheme over a quasi-regular grid is $C^{1+\mu}$ for any

$$\mu \in (0, -\frac{1}{\tilde{K_2}+1} \log_2(\tilde{\mu}_2)) \tag{1.84}$$

$$\begin{aligned} \mu &\in (0, -\frac{1}{3} \log_2(3/4)) \\ &= (0, 0.1383) \end{aligned} \tag{1.85}$$

It is pointed out in [15] that (1.28) gives a bound which is not optimal, hence we do not expect the estimate in (1.85) to be optimal either. They do, however, give an estimate which is independent of the how the quasi-regular grid is constructed since a quasi-regular grid is always equivalent to a regular grid. We mentioned that a quasi-regular grid is homogeneous, and for the homogeneous and dyadically balanced grid where $\frac{1}{2} \leq \lambda \leq \lambda_0$ we already stated that the regularity estimate is stronger than (1.85). In [4] it was conjectured that it the scheme will produce a $C^{2-\epsilon}$ limit function whenever $\lambda < 1$, while [11] showed that at least for $\lambda \leq 0.7142$ this regularity is maintained. However, the conjecture was proven wrong by Floater [10], who showed that one upper bound for λ is 0.8847 by constructing a counterexample.

1.5 Local reduction to a stationary scheme

In the article by Warren [24] necessary and sufficient conditions for C^k continuous limit curves of a wide class of stationary subdivision schemes are presented. As well as the method of local reduction of a non-stationary scheme based on semi-regular multi-level grid to a stationary scheme. The main focus of our short review is to recover and understand his results and analysis of the four-point subdivision scheme over a semi-regular multi-level grid. Spectral properties of a finite submatrix of a stationary subdivision matrix are used in order to show that the scheme produces a C^1 limit curve also in the semi-regular case. Note that we do not discuss sufficient conditions for C^k continuity of the limit function, only necessary, due to the limited scope of this review. Whenever we use that a function is C^k continuous, it is assumed to be given as a result in the article. The limit function can be expressed as

$$g(x) = \lim_{j\to\infty} g_j(x), \tag{1.86}$$

where $g_j(x)$ was defined in the section on convergence. For a stationary scheme where $S_j = S \,\forall\, j \geq 0$, it might be tempting to express the limit function as a linear combination of limit functions induced by eigenvectors of S. But S is assumed bi-infinite and can be akward to relate to, especially in terms of the existence of eigenvalues and eigenvectors as the eigenpairs do not generally exist. As mentioned, our subdivision method is local. Every new point only depends on a finite number of old, so a point on the limit curve only depend on a finite number of original control points. For example, in $[-1, 1]$ the number of original control points affecting the limit function is at most $2n_S + 1$. For the four point scheme this is illustrated in Fig 1.4. 7 inital control points are the only original control points influencing the limit function in $[-1, 1]$, and only 7 old points govern 7 new points. This implies that to study local properties of the limit curve we do not need to study the assumed infinite vector of control points, nor the infinite subdivision matrix at each level, but only a finite dimensional portions of the original subdivision matrix.

Example 1.4. *For our reappering example, we have the following local linear system.*

$$\bar{\mathbf{f}}_{j+1} = \bar{S}\bar{\mathbf{f}}_j$$

$$\begin{pmatrix} f_{j+1,2k-3} \\ f_{j+1,2k-2} \\ f_{j+1,2k-1} \\ f_{j+1,2k} \\ f_{j+1,2k+1} \\ f_{j+1,2k+2} \\ f_{j+1,2k+3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{pmatrix} \begin{pmatrix} f_{j,k-3} \\ f_{j,k-2} \\ f_{j,k-1} \\ f_{j,k} \\ f_{j,k+1} \\ f_{j,k+2} \\ f_{j,k+3} \end{pmatrix}$$

The local submatrix $\bar{S}$ of S is the 7×7 submatrix shown above.

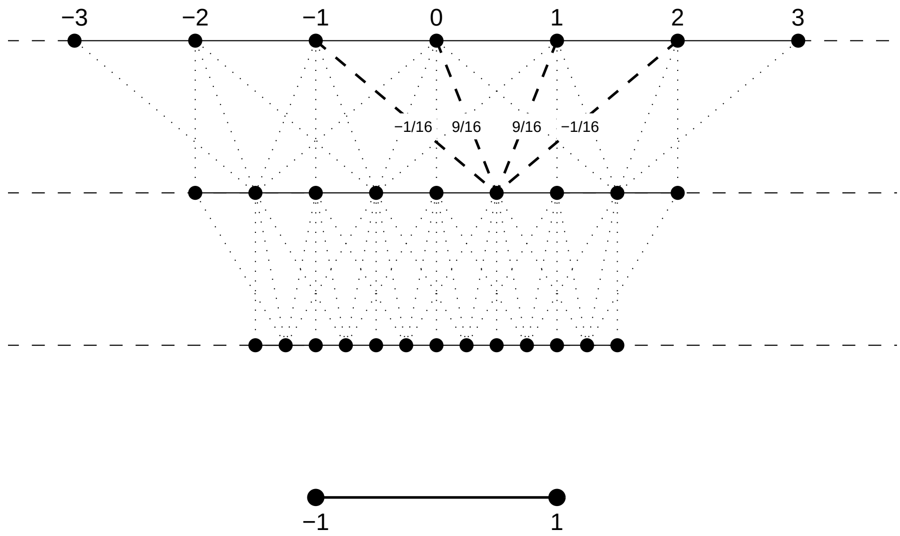

Figure 1.4: Locality illustration

Lets first recap the general setting in terms of subdivision over semi-regular multi-level grids. We are given a strictly increasing irregular sequence $X_0 = x_{0,i}$ and define the refined sequences as midpoint insertions on the previous.

$$x_{j+1,2i} = x_{j,i} \tag{1.87}$$

$$x_{j+1,2i+1} = \frac{x_{j,i} + x_{j,i+1}}{2} \tag{1.88}$$

This is what is defined in this thesis, as well as in other works, e.g [4], as a semi-regular multi-level grid. We introduced the bi-infinite matrix notation to describe the subdivision process. If the initial control points are given $\mathbf{f} = \mathbf{f}_0$ associated with the initial arbitrary sequence, then the new points are generated by an application of the subdivision matrix to the old points.

$$\mathbf{f}_{j+1} = S[X_j]\mathbf{f}_j$$

Here the notation $S[X_j]$ is used to emphasis that the matrix is dependent of the parametrization of level j. The intermediate subdivision curve is viewed as the linear interpolant to $(X_j, \mathbf{f}_j)$, here denoted $L[X_j, \mathbf{f}_j]$, and we view the limit function $F[X, \mathbf{f}_0]$ as a pointwise limit of the linear interpolants.

$$F[X, \mathbf{f}_0] = \lim_{j\to\infty} L[X_j, \mathbf{f}_j](t)$$

F is a linear operator and by linearity we can express the limit function as a linear combination of F applied to the infinite unit vectors $\mathbf{e}_i$.

$$F[X_j, \mathbf{f}_j](t) = \sum_i \mathbf{f}_{j,i} F[X_j, \mathbf{e}_i](t)$$

Theorem 1 of the article shows that the support of $F[X_j, \mathbf{e}_i]$ is $[x_{j,i-n_S}, x_{j,i+n_S}]$, where n_S is defined as in the first section. $F[X_j, \mathbf{e}_i]$ are refered to as the basis functions. Then we can find local finite dimensional matrix $\bar{S}[X_j] \in \mathbb{R}^{2n_S+1 \times 2n_S+1}$, relating $2n_S + 1$ old control point to $2n_S + 1$ new ones. But as we see, these matrices are dependent on the refinement level, hence the spectral properties vary from level to level. We wish to be able to write our limit function locally as

$$F[X_0, \mathbf{f}_0] = \sum_{i=0}^{2n_S+1} \mathbf{f}_{0,i} F[X_0, \mathbf{v}_i], \tag{1.89}$$

where $\mathbf{v}_i$ are eigenvectors of a single subdivision matrix $\bar{S}$, as we would in the stationary case. To reduce this non-stationary subdivision scheme locally to stationary scheme we look at the limit function in a small neighborhood of a given initial parameter. We assume that $x_{0,0} = 0$[6] and take this as our initial parameter. Then observe that by construction all the parameters from $x_{j,-2^j}$ to $x_{j,0}$ are equally spaced, and likewise all the parameters from $x_{j,0}$ to $x_{j,2^j}$ are as well, see Fig 1.5. Theorem 2 tells us that we can find some refinement level j and another parameter sequence $\hat{X}$ constructed by the parameter sequence on level j to be regular on each side of zero such thats the limit functions generated by $\hat{X}$ and X_j respectively, agree on the interval $(x_{j,-1}, x_{j,1})$:

$$F[X_j, \mathbf{f}_j](x) = F[\hat{X}_0, \mathbf{f}_j](x) \tag{1.90}$$

The author explains that by using a parameter sequence on the form of $\hat{X}$ as the initial parameter sequence, namely[7]

$$\hat{x}_{i,0} = |x_{j,-1}| i \;\forall\; i < 0 \tag{1.91}$$

$$\hat{x}_{i,0} = (x_{j,1}) i \;\forall\; \geq 0 \tag{1.92}$$

and then apply dyadic refinement to this sequence, yields a stationary scheme.

$$\hat{X}_j = \frac{\hat{X}_0}{2^j},$$

This is due to the fact that $S[\hat{X}_j] = S[\frac{\hat{X}_0}{2^j}] = S[\hat{X}_0]$. The limit function can be expressed as a linear combination of limit functions where the control points are either

[6]For simplicity, and without loss of generality, since all other cases can be dealt with by re-indexing

[7]In the article the first equation (1.91) differs from mine, I used absolute value for the parameters to be negative left of 0

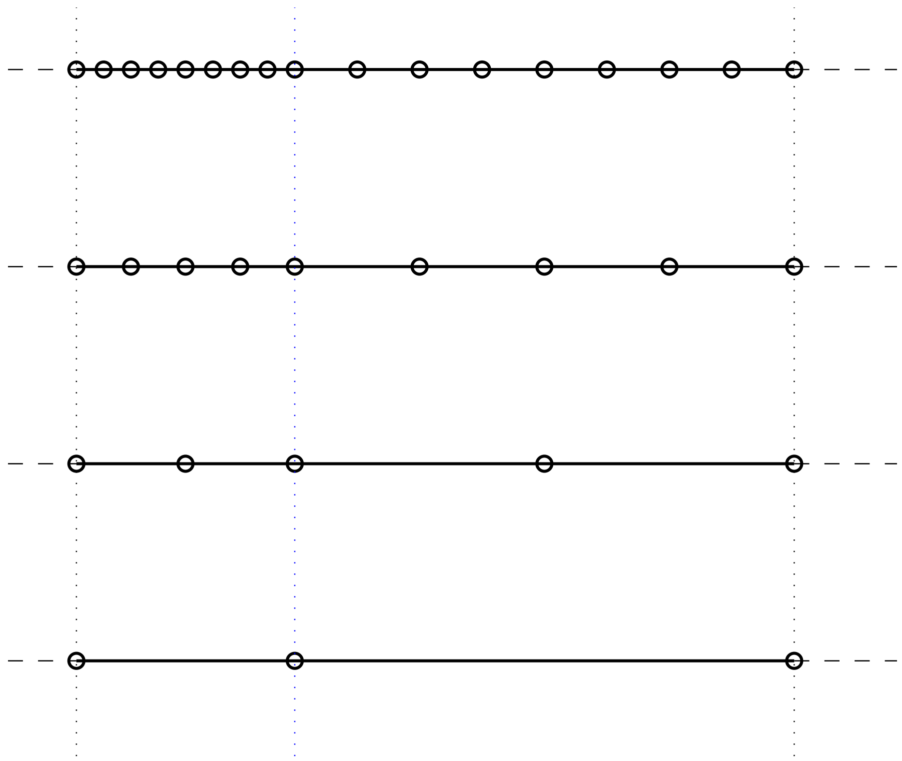

Figure 1.5: Parameters around the origin

the eigenvectors, or possibly the generalized eigenvectors $S\mathbf{v}_i = \lambda_i\mathbf{v}_i + \mathbf{v}_{i-1}$, of the the finite dimensional submatrix of S. Further the limit function induced by an eigenvectors satisfies an important relation, if $S\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v}$ eigenvector, λ eigenvalue of S

$$\lambda F[X, \mathbf{v}](t) = F[X, \mathbf{v}]\left(\frac{t}{2}\right).$$

For a generalized eigenvector the corresponding relation is

$$\lambda F[X, \mathbf{v}_i](t) = F[X, \mathbf{v}_i]\left(\frac{t}{2}\right) + F[X, \mathbf{v}_{i-1}](t).$$

We have uncovered that the limit function depend linearly on the limit functions induced by an eigenvector or generalized eigenvector, hence the smoothness analysis can be restricted to analysis of these limit functions. The dilation relations above are used to determine the smoothness of these.

Analysis of the semi-regular four-point scheme

The finite 7×7 submatrix associated with the four-point scheme is given as

$$S = \begin{pmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{pmatrix}.$$

The matrix does not have a full span of eigenvectors, it has the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_6 = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, -\frac{1}{6}, -\frac{1}{6}$. The eigenvector $\mathbf{v}_0$ is $(1, 1, \ldots, 1)^T$, while $\mathbf{v}_1 = \alpha(0, 1, 2, 3, 4, 5, 6)^T$ for some constant $\alpha \in \mathbb{R}$, showing the reproduction of constant and linear functions respectively. The following two theorems stating necessary conditions for C^k continuity of the basic limit functions will help us deduce that the four-point scheme is not C^2.

Theorem 1.4. *Let $S\mathbf{v} = \lambda\mathbf{v}$ with $|\lambda| \geq \frac{1}{2^k}$. If $F[X_0, \mathbf{v}](t) \in C^k$ and $F[X_0, \mathbf{v}] \neq 0$ there exist i, $0 \leq i \leq k$ such that $\lambda = \frac{1}{2^i}$ and $F[X_0, \mathbf{v}](t) = c_i t^i$ for some $c_i \neq 0$.*

The proof of this theorem is given in [24]. The next theorem is concerned with necessary conditions for a generalized eigenvector to produce a C^k limit function.

Theorem 1.5. *Let $S\mathbf{v}_j = \lambda_j\mathbf{v}_j + \mathbf{v}_{j-1}$. If $F[X_0, \mathbf{v}_j](t) \in C^k$ and $F[X_0, \mathbf{v}_j] \neq 0$ there exist i, $0 \leq i < k$, such that $\lambda = \frac{1}{2^i}$ and $F[X_0, \mathbf{v}_j](t) = c_i t^i$ for some $c_i \neq 0$*

Proof. Assume $F[X_0, \mathbf{v}_{j-1}](t), F[X_0, \mathbf{v}_j](t) \in C^k$ and both not identically zero, with $\lambda_j = \frac{1}{2^k}$. Then by Theorem 1.4

$$F[X_0, \mathbf{v}_{j-1}](t) = Ct^k \qquad \text{and} \qquad F^{(k)}[X_0, \mathbf{v}_{j-1}](t) = C'$$

Hence

$$2^k \lambda_j F^{(k)}[X_0, \mathbf{v}_j](t) + 2^k F^{(k)}[X_0, \mathbf{v}_{j-1}] = F^{(k)}[X_0, \mathbf{v}_j](t/2)$$
$$F^{(k)}[X_0, \mathbf{v}_j](t) + C'' = F^{(k)}[X_0, \mathbf{v}_j](t/2)$$

showing that $F^{(k)}[X_0, \mathbf{v}_j]$ diverge as $t \to \infty$, contradicting $F[X_0, \mathbf{v}_j] \in C^k$ □

Example 1.5. *For the four-point scheme we have that $\lambda := \lambda_2 = \lambda_3 = \frac{1}{4}$. So we assume, to get a contradition that $\mathbf{v}_2$, the eigenvector associated with λ_2, produce a quadratic limit function by Theorem 1.4. For the generalized eigenvector $\mathbf{v}_3$ we have that*

$$S\mathbf{v}_3 = \lambda \mathbf{v}_3 + \mathbf{v}_2$$

In terms of limit functions, this implies

$$\frac{1}{4}F[X_0, \mathbf{v}_3](t) + F[X_0, \mathbf{v}_2](t) = F[X_0, \mathbf{v}_3](t/2)\frac{1}{4}F[X_0, \mathbf{v}_3](t) + c_2 t^2 = F[X_0, \mathbf{v}_3](t/2)$$

Taking the second derivative of the expression

$$F^{(2)}[X_0, \mathbf{v}_3](t) + 2c_2 = F^{(2)}[X_0, \mathbf{v}_3](t/2)$$

which diverge as $t \to 0$, hence $F[X_0, \mathbf{v}_3] \notin C^2$. Therefor the general limit curve cannot not be C^2 either, since it depends linearly on $F[X_0, \mathbf{v}_3]$. For a semi-regular case, we have already argued that we can find j and a new parameter sequence $\hat{X}$ as stated previously such that the limit functions resulting from using the two different parameter sequences $\mathbf{X}, \hat{\mathbf{X}}$ agree locally. $\hat{\mathbf{X}}$ can be written on the form $\dots, -3, -2, -1, 0, c, 2c, 3c, \dots$ where $c = \frac{x_{j,1}}{x_{j,-1}}$. Now the local subdivision matrix $\hat{S}$ takes the form of

$$\hat{S} = \begin{pmatrix} -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(1+2c)}{16(2+c)} & \frac{3(1+2c)}{8(1+c)} & \frac{3}{8} + \frac{3}{16c} & -\frac{3}{8c(2+3+c+c^2)} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3c^3}{8(1+3c+3c^2)} & \frac{3(2+c)}{16} & \frac{3(2+c)}{8(1+c)} & -\frac{-(2+c)}{16(1+2c)} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{16} & \frac{9}{16} & \frac{9}{16} & -\frac{1}{16} \end{pmatrix}.$$

It can be shown that $\hat{S}$ have eigenvalues independent of c, $(\lambda_0, \lambda_1, \dots, \lambda_6) = (1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, -\frac{1}{6}, -\frac{1}{6})$ and the same eigenvectors as for the stationary scheme, hence by precisely the same arguments as above, the general limit function is strictly C^1.

1.6 Final remarks

In this chapter we have tried to introduced binary subdivision with emphasis on how smoothness results depend on the underlying parametrization at each subdivision level.

Without going into immense detail we have tried to introduce two methods of analysis, spectral analysis and derived schemes, through examples. We have all along assumed that the subdivision rules for all the points are derived similarly, as weighted means of the old points, the masks differ from refinement level to the next, but the logic of the construction is the same. We have also assumed that we only refined with respect to control points, and we have neglected the behaviour at the end of the limit function. As we saw in the four-point scheme, all but the first two and the last two values where interpolated. A suggestion on how to avoid using four additional points are given in the next chapter. The subdivision schemes we consider will we local, affine and bounded, but will not only depend on the function values. We will be given additional information in terms of derivatives of some order at the first point. There is literature and research concering what is known as Hermite subdivision, for an introduction see e.g [8, 7]. Hermite subdivision is subdivision on both given derivatives and function values, but then the derivatives are usually given at each position. We can say that the cases we consider are a mix of Hermite and ordinary univariate subdivision.

2 Boundary conditions for subdivision

2.1 Introduction

As discussed earlier, the four point scheme is defined with initial values f_i, where the support is $i = -2, -1, \ldots, n+2$, whereas the limit function is only defined over $[x_0, x_n]$. The four additional points, influence the limit function, but is not interpolated. One way to control the behaviour of the limit curve at the support ends is to specify a prescribed relationship between the initial control points. Another is to supply end derivatives and somehow use this to change the subdivision rules at the ends. We will consider the latter of the two, but provide an example on the first through an example given in [26]. Based on the requirements in [19] we can summarize some of the different kinds of end conditions that can be useful to consider in order to control the behavior of the curve at the end of the interval. Among them are:

Prescribed derivatives at the endpoints: Adapt the scheme to make sure that the limit function has the same value as the prescribed derivatives up to some degree at the endpoints.

Natural end-condition: We can ensure that the second derivative of the curve at the endpoints is zero.

Constant curvature: We can ensure that the second derivative is constant at each of the endpoints.

In the following section we will suggest a general subdivision method for univariate interpolation when both function values and one or more derivatives are given at the first value[1]. This will answer to the first endpoint condition on our list. The resulting schemes can be viewed as modifications of the Dubuc-Deslauriers schemes and an interesting extension would be to investigate how the smoothness of these schemes is related to the original schemes. We will in the next chapter, however, only determine a bound on the smoothness of the interpolant for a special case of this method. We will refer to and define this as *the cubic case*. Next we will introduce a natural extension of this scheme to a tensor product scheme. Finally, for completeness, we will review related work on the area. The references here will be to the Ph.D-thesis of Adi Levin [14] and to an article by Cai Zhijie [26].

[1] Symmetry in the rules makes sure that we can adapt this to the right-most endpoint as well, but analysis are similar so we omit considering the end condition at the right end of the interval.

2.2 A boundary condition scheme for interpolatory subdivision

In this section we suggest a general subdivision method where the initial values are given both as function values and a general numbers of derivatives at the first point. Assume that we are given a finitely supported set of values

$$\{\mathbf{f} = \{f_k = f(x_k)\}_k |\ f_k \in \mathbb{R}\ \forall k \in \mathbb{Z},\ f_k \neq 0,\ k \in \{0, 1, \ldots, N\}\},$$

associated with a strictly increasing sequence of points

$$X_0 = \{x_k\}_{k \in \mathbb{Z}}.$$

In addition, suppose that we are given the values of the derivatives of the function f at x_0 up to some order $s \geq 0$, which we denote $m_0^{(l)} := f^{(l)}(x_0)$, $l = 1, \ldots s$. We seek a smooth interpolant g such that

$$\begin{aligned} g(x_k) &= f_k, & k &= 0, \ldots, n, \\ g^{(l)}(x_0) &= m_0^{(l)}, & l &= 1 \ldots s, \end{aligned}$$

where $n \leq N$ is the largest initial index for which the interpolant is well-defined[2]. Initialise by setting $f_{0,k} = f_k$, $\forall\ k$. For the scheme to be interpolatory we require that

$$\begin{aligned} f_{j+1,2k} &= f_{j,k}, & &\forall k \\ m_{j+1}^{(l)} &= m_j^{(l)}, & &l = 1, \ldots, s, \end{aligned} \tag{2.1}$$

where j denotes the current refinement level. The new odd point $f_{j+1,2k+1}$ is determined using a local polynomial interpolant to neighbouring data symmetric about the new odd parameter $x_{j+1,2k+1}$. The subdivision process is initialised by setting

$$\begin{aligned} f_{0,k} &= f_k, \\ m_{0,0}^{(l)} &= m_0^{(l)} \text{ for } l = 1, \ldots, s, \\ x_{0,k} &= x_k \text{ for } k = 0, \ldots, N. \end{aligned}$$

We need to find the degree d of the local interpolant polynomial. We find the degree d by looking at the number of conditions we have to the left of the first odd point $x_{1,1}$, namely $s+1$, s derivative values and one function value. Hence we need the same number of conditions to the right, since we required symmetry. Then $d + 1 = 2s + 2$. In the cubic case this selection is illustrated for the first two odd points in Figure 2.1. If we want to construct a cubic interpolant we require that we are given one derivative value. If we are provided with no derivative values the only choice we have is to construct a linear interpolant. The local interpolant might be osculatory[3], i.e interpolate derivative

[2] This index n will be elaborated shortly

[3] A concept we will define further shortly.

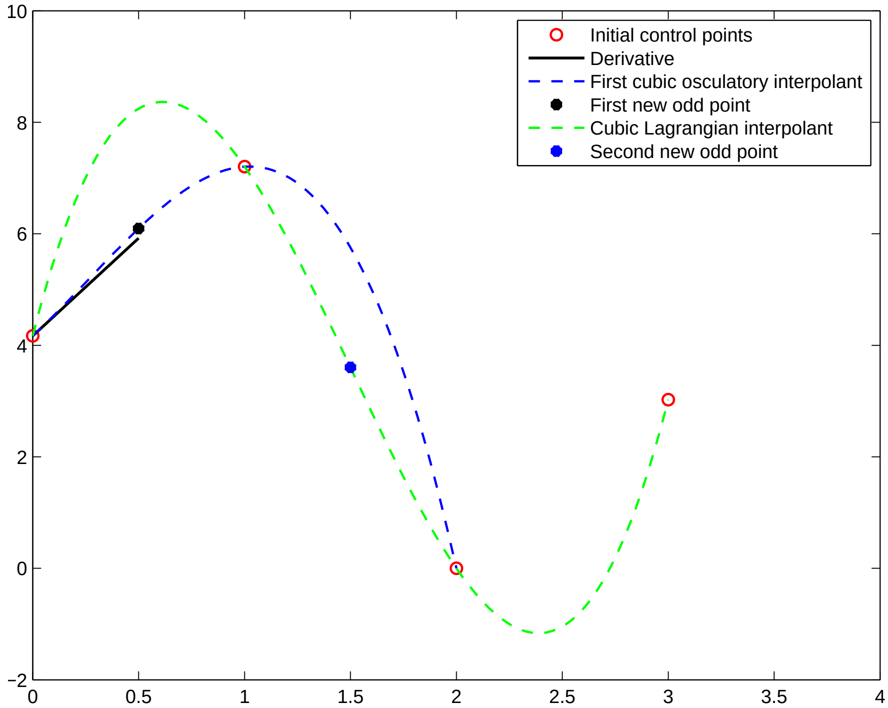

Figure 2.1: Illustration of selection of new control points in the cubic case

values, or Lagrangian, depending on which point we want to calculate. The last odd value can be determined in the first iteration and depend on $f_{0,N-d}, f_{0,N-d-1}, \dots, f_{0,N}$. By this it follows that $n = N - \frac{d+1}{2} = N - s - 1$. For the cubic case this is illustrated in Figure 2.2, where we see how the last grid point grid converge to the initial index $N-2$ as j increases. In the following section we will state the subdivision scheme for a general number of derivatives s and an associated degree d. We will focus mainly of the case where $s = 1, d = 3$ since we are to prove convergence and regularity results for this case later. We will refer to this as the *the cubic case*. Recall that the degree of our local interpolant, d, is uniquely determined through the given number of derivatives s as $d = 2s+1$, and as mentioned above we will consider two types of interpolants; Lagrangian and osculatory. We will require familiarity with ordinary Lagrangian interpolation, but briefly introduce osculatory interpolation as well as how the solution to an osculatory interpolation problem can be viewed as a natural extension of the Newton form for Lagrangian interpolation.

General polynomial interpolation

A Hermite-Birkhoff interpolation problem is the most general definition of a polynomial interpolation problem of a general degree d. As described in [21], it can be stated quite compactly through an incidence matrix $E = \{\{\epsilon_{i,j} \in \{0,1\}\}^{(n+1)\times(d+1)}\}$, associated

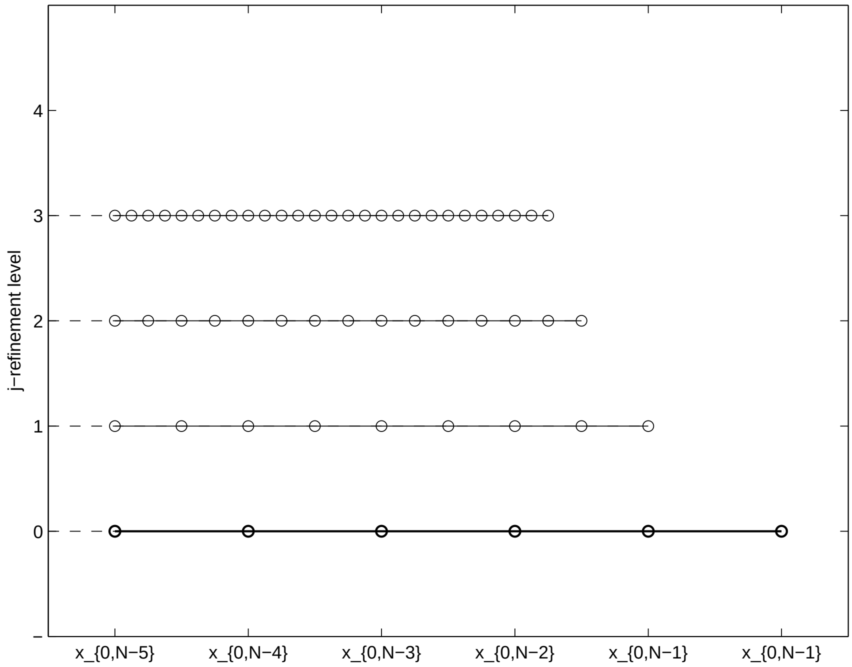

Figure 2.2: Illustration of refinement grid at the end of the parameter interval

with the set of ordered pairs $e = \{(k,i) \mid \epsilon_{k,i} = 1\}$, with the property that $|e| = d+1$. Assuming that we have $x_0 < x_1 < \ldots < x_n$ and that no row consist entirely of zeros. Then the incidence matrix describes the interpolation problem of finding a d degree polynomial p satisfying $d+1$ interpolation conditions. In other words, find $p \in \pi_d$ such that

$$p^{(i)}(x_k) = f^{(i)}(x_k),\ \forall\ (k,i)\ \in e, \tag{2.2}$$

where $f^{(i)}(x_k)$ are the prescribed data, in terms of function values and derivatives. The superscript indicated the order of derivative, where $f^{(0)}(x_k) = f(x_k)$. As we can see this is a very general problem definition. We can supply any number of derivatives and any number function values at every x_k. In general this problem is not always poised. We have the following definition of a poised Hermite-Birkhoff problem.

Definition 2.1 (Poised Herminte-Birkhoff problem [21])**.** *A Hermite-Birkhoff problem is poised, provided that if*

$$p(x) \in \pi_d \qquad \textit{and} \tag{2.3}$$
$$p^{(i)}(x_k) = 0, \qquad \forall\ \ (k,i)\ \in e, \tag{2.4}$$

then $p(x) \equiv 0$.

Example 2.1. *Recall that for the general subdivision scheme introduced above, the first local interpolant we consider is the problem of finding $p \in \pi_d$ such that*

$$\begin{aligned} p^{(i)}(x_0) &= f^{(i)}(x_0), \qquad && i \in \{1,\dots,r\},\ r \le s, \\ p(x_k) &= f(x_k), && k \in \{0,\dots,(2s+1)-r\} \end{aligned} \tag{2.5}$$

A total of $d+1 = 2s+2$ conditions and we also have $d+1$ unknown coefficients. So the problem has a unique solution for any data if and only if the associated (2.2) is poised. The indices matrix $E \in \{0,1\}^{((2s+1)-r)\times(d+1)}$ is given as

$$E = \begin{pmatrix} 1 & 1 & 1 & \dots \\ 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{2.6}$$

Also $e = \{(0,0),(0,1),\dots,(0,r),(1,0),(2,0),\dots,(2s+2-r,0)\}$. Assume, to get a contradiction, that we have two distinct solutions to (2.7), $q, \hat{q} \in \pi_d$. Then $p = q - \hat{q} \neq 0$ is a solution to (2.5) with the zero data,

$$p^{(i)}(x_0) = 0, \qquad i = 1,\dots,r,\ r \le s, \tag{2.7}$$

$$p(x_k) = 0, \qquad k = 0,\dots,(2s+1)-r \tag{2.8}$$

Then x_0 is a zero of multiplicity $r+1$, while $x_k,\ k = 1, 2s+1-r$ are all simple zeroes. Hence $p(x)$ is a polynomial of d degree with $d+1 = 2s+2$ zeroes counting multiplicities. Thus $p \equiv 0 \Leftrightarrow q = \hat{q}$, proving that the solution is unique.

All interpolation problems that we encounter in this thesis will be poised.

Newton's interpolation formula and divided differences

In this section we will briefly introduce Newton's interpolation formula for an osculatory interpolant and with it the notion of general divided differences over multiple points. Let $\{x_i\}_{i=0}^{k}$ be a sequence of distinct real values. Assume we are provided with samples of some real valued function f at x_i and derivatives up to some $c_i - 1$ at each x_i. We want to find a polynomial p of degree d where $d + 1 = \sum_{i=0}^{k} c_i$. such that

$$p^{(j)}(x_i) = f^{(j)}(x_i), \qquad j \in \{0,\dots,c_i-1\} \text{ and } i \in \{0,\dots,k\} \tag{2.9}$$

By the same arguments as in Example 2.1 above this problem is uniquely solvable. Assume $f \in C^{\max_i(c_i-1)}[x_0, x_n]$ and introduce

$$\{z_i\}_{i=0}^{d} = (\underbrace{x_0,\dots,x_0}_{c_0},\dots,\underbrace{x_i,\dots,x_i}_{c_i},\dots,\underbrace{x_k,\dots,x_k}_{c_k})$$

Then we propose that the solution, $p \in \pi_d$, to the problem 2.9 can be expressed as

$$p(x) = \sum_{i=0}^{d} [z_0, \dots, z_i] f \phi_i(x), \tag{2.10}$$

where $\phi_0(x) = 1$, while $\phi_i(x) = (x - z_{i-1})\phi_{i-1}(x)$. $[z_0, \dots, z_i]f$ is the i^{th} order divided difference to f over the values $z_0, \dots, z_i$ is defined recursively,

$$[z_k, \dots, z_i]f = \begin{cases} \frac{[z_{i+1},\dots,z_k]f - [z_i,\dots,z_{k-1}]f}{z_k - z_i} \text{ if } & z_i \neq z_k \\ \\ \frac{f^{(k-i)}(z_i)}{(k-i)!} \text{ if } & z_k = z_i \end{cases} \tag{2.11}$$

Here $[z_0]f = f(z_0)$, $f^{(0)}(z_i) = f(z_i)$ [12]. This is due to the fact that the ordinary divided differences can be shown to be continuous in its arguments. For a detailed proof of the form of the osculatory interpolation formula above we refeer the reader to [20]. In the formulation of our schemes we will have to create local interpolants which depend on which refinement level we are on, so the in following redefinitions of divided differences we recall our notation for the function values at refinement level j, $f_{j,k}$.

Definition 2.2 (Divided differences over non-consequtive points). *Set $[i]f_{j,k} = f_{j,k+i}$. Then for any distinct integers*

$$i_0, i_1, \dots, i_m,$$

let $[i_0, i_1, \dots, i_m]f_{j,k}$ denote the m^{th} divided difference of the values $f_{j,k+i_0}, \dots, f_{j,k+i_m}$ at $x_{k+i_0}, \dots, x_{k+i_m}$, defined by the recurrence relation

$$[i_0, i_1, \dots, i_m]f_{j,k} = \frac{[i_1, i_2, \dots, i_m]f_{j,k} - [i_0, i_1, \dots, i_{m-1}]f_{j,k}}{x_{j,k+i_m} - x_{j,k+i_0}} \tag{2.12}$$

Analogous to (2.11), given s, $0 < s < d+1$ derivative values at, say, f_{k+i_0} we can allow $i_0 = i_1 = \ldots = i_s$ as long as $i_{s+1} \neq i_0 \neq i_m$ and similarly denote the divided difference of order m as $[i_0, i_1, \dots, i_m]f_k$ even if some of the indices coincide. Yet another notation for divided differences will be used. Let $f_{j,k}^{[0]} = f_{j,k}$, then we have the following recursive definition.

$$f_{j,k}^{[m]} = \frac{f_{j,k+1}^{[m-1]} - f_{j,k}^{[m-1]}}{h_{j,k}^{[m]}}, \tag{2.13}$$

where $h_{j,k}^{[m]} = x_{j,k+m} - x_{j,k}$. We will use all three definitions, (2.11), (2.12) and (2.13) when refering to divided differences. Eventually we will need notation for differences of divided differences of general order.

$$\tilde{f}_{j,k}^{[m]} := f_{j,k+1}^{[m-1]} - f_{j,k}^{[m-1]} = h_{j,k}^{[m]} f_{j,k}^{[m]} \tag{2.14}$$

We will also use that the leading coefficient of any interpolation polynomial is unique. The leading coefficient is given as $[z_{i_0}, z_{i_1}, \ldots, z_{i_d}]f$, for any permutation of the integers i_l are distinct since the divided difference are symmetric in its arguments. Since a interpolation polynomial of degree d is uniquely determined by $d+1$ conditions, then if we are given more that $d+1$ conditions, say $d+1+m$ we can always chose a subset of $d+1$ of these for expressing the polynomial, and hence also the leading coefficient.

2.2.1 Formulation of the scheme

Now we are almost ready to state the formulation of the scheme for general degree. But before that we need to introduce some additional notation. Let $p^{[d]}_{j,k+s}(x)$ denote the unique polynomial of degree $\leq d$ that interpolates $f_{i,j}$ at $x_{i,j}$, $i = k, k+1, \ldots, k+d$. In the Newton basis this takes the following general form

$$p^{[d]}_{j,k}(x) = \sum_{i=0}^{d} [z_{j,0}, \ldots, z_{j,i-1}] f_{j,k-1} \phi_i(x), \tag{2.15}$$

$$\{z_{j,i}\}_{i=0}^{d} = (x_{j,k}, x_{k+1}, \ldots, x_{j,k+d}) \tag{2.16}$$

where $\phi_0(x) = 1$, and $\phi_i = (x - z_{j,i-1})\phi_{i-1}(x)$. Likewise we let $p^{[d]}_{j,s-r}$ denote the unique osculatory interpolant to the data m^k_0, where $k = r, \ldots, 1$ at $x_{j,0}$ and the function values $f_{j,i}$ associated with $x_{i,j}$ for $i \in \{0, 1, \ldots, d-r\}$. By concepts we introduced in the previous section, this can be expressed in Newton form quite similarly.

$$p^{[d]}_{j,s-r}(x) = \sum_{i=0}^{d} [z_{j,0}, \ldots, z_{j,i-1}] f_{j,k} \phi_i(x) \tag{2.17}$$

$$\{z_{j,i}\}_{i=0}^{d} = (\underbrace{x_{j,0}, \ldots, x_{j,0}}_{r+1}, x_1, \ldots, x_{j,d-r}) \tag{2.18}$$

Then we express our general interpolatory subdivision scheme as

$$\begin{aligned} f_{j+1,2k} &= f_{j,k} \\ f_{j+1,2k+1} &= p^{[d]}_{j,k}(x_{j+1,2k+1}) \end{aligned} \tag{2.19}$$

More precisely, to demonstrate the definitions above, for the cubic case where $s = 1$ and $d = 3$ we have:

$$\begin{aligned} f_{j+1,0} &= f_{j,0} \\ f_{j+1,1} &= p^{[3]}_{j,0}(x_{j+1,1}) \\ f_{j+1,2k} &= f_{j,k} \\ f_{j+1,2k+1} &= p^{[3]}_{j,k}(x_{j+1,2k+1}), k > 0, \end{aligned}$$

where $p_{j,0}^{[3]}(x)$ is the unique osculatory interpolant to $m_0^1, f_{j,0}, f_{j,1}, f_{j,2}$, with the explicit formula

$$\begin{aligned} p_{j,0}^{[3]}(x) &= f_{j,0} + [x_0, x_0] f_{j,k}(x - x_0) \\ &+ [x_0, x_0, x_1] f_{j,k}(x - x_0)^2 \\ &+ [x_0, x_0, x_1, x_2] f_{j,k}(x - x_0)^2 (x - x_1) \end{aligned} \tag{2.20}$$

Furthermore $p_{j,k}^{[3]}$ for $k > 0$ is the unique Lagrangian interpolant to $f_{j,k-1}, f_{j,k}, f_{j,k+1}, f_{j,k+2}$, given as

$$\begin{aligned} p_{j,k}^{[3]}(x) &= f_{j,k-1} + [x_{k-1}, x_k] f_{j,k}(x - x_{k-1}) \\ &+ [x_{k-1}, x_k, x_{k+1}] f_{j,k}(x - x_{k-1})(x - x_k) \\ &+ [x_{k-1}, x_k, x_{k+1}, x_{k+2}] f_{j,k}(x - x_{k-1})(x - x_k)(x - x_{k+1}) \end{aligned} \tag{2.21}$$

On the *regular* multi-level grid the cubic scheme $s = 1, d = 3$ yield

$$\begin{aligned} f_{j+1,0} &= f_{j,0} \\ f_{j+1,1} &= \frac{1}{32}(21 f_{j,0} + 6 m_0^1 (x_{j,1} - x_{j,0}) + 12 f_{j,1} - f_{j,2}) \\ f_{j+1,2k} &= f_{j,k}, k > 1 \\ f_{j+1,2k+1} &= -\frac{1}{16} f_{j,k-1} + \frac{9}{16} f_{j,k} + \frac{9}{16} f_{j,k+1} - \frac{1}{16} f_{j,k+2}, k > 1 \end{aligned} \tag{2.22}$$

For the quintic case $s = 2, d = 5$ we have

$$\begin{aligned} f_{j+1,0} &= f_{j,0}, \\ f_{j+1,1} &= p_{j,0}^{[5]}(x_{j+1,1}), \\ f_{j+1,2} &= f_{j,1}, \\ f_{j+1,3} &= p_{j,1}^{[5]}(x_{j+1,3}), \\ f_{j+1,2k} &= f_{j,k}, k > 2, \\ f_{j+1,2k+1} &= p_{j,k}^{[5]}(x_{j+1,2k+1}), k > 2. \end{aligned}$$

Here $p_{j,0}^{[5]}(x)$ is the osculatory interpolant to $f''(x_0) = m_0^2, f'(x_0) = m_0^1, f_{j,0}, f_{j,1}, f_{j,2}, f_{j,3}$, and $p_{j,1}^{[5]}(x)$ is the osculatory interpolant to $m_0^1, f_{j,0}, f_{j,1}, f_{j,2}, f_{j,3}, f_{j,4}$, while as usual $p_{j,k}$ is the Lagrange interpolant to $f_{j,k-2}, f_{j,k-1}, f_{j,k}, f_{j,k+1}, f_{j,k+2}, f_{j,k+3}$. Since all the rules are generated by local interpolation it follows that the scheme is local. The concept of locality is somehow different from the one we introduced in the first chapther. Here the new values depend on a finite number of old points, in addition it may depend on derivative values as well. The schemes are all bounded since the initial values are assumed to be bounded, hence the coefficients of the interpolation polynomials will be bounded. We can also state that the scheme is affine, in the sense that that the for the constant function, then all the derivatives are zero. It is worth pointing out that now we not longer talk about masks or stensils, but we still use the same logic as in the general setting. New points are calculated using a finite number of old points.

The schemes we have suggest is interpolatory by construction and uses local interpolants, but we have no guarantee for the smoothness of these schemes. This might be a topic for future research. However, as the number of given derivatives increases, the applicability of the scheme is not really strengthned, since it might not be natural to "invent" high order derivative values to be interpolated. We consider the most interesting extension to be the case where $s = 2, d = 5$, as we know that the Dubuc-Deslauriers of order 5 is C^2. In addition it can be useful to specify and control the second derivative for shape preserving properties of the curve.

Now that we have suggested a general subdivision scheme, we will use the next section to go into more depth in the case where $s = 1, d = 3$.

2.3 The cubic case

Given a finite sequence of values $\{f_i\}_{i=0}^N$ associated with a sequence of parameters $\{x_i\}_{i=0}^N$ and a value for the derivative at x_0, $f_0' = m_0$, we seek an interpolant $g : [x_0, x_n]$, where $n = N - 2$ such that

$$\begin{aligned} g(x_k) &= f_k,\ k = 0, \ldots, n, \\ g'(x_0) &= m_0. \end{aligned} \tag{2.23}$$

Initialise by setting $g_{0,j} = f_j$ for $j \in \{0, 1, \ldots, N\}$, and $m_{0,0} = m_0$. The scheme then takes the following form

$$\begin{aligned} g_{j+1,0} &= g_{j,0}, \\ g_{j+1,1} &= p_{j,0}^{[3]}(x_{j+1,1}), \\ g_{j+1,2k} &= g_{j,k}, k > 1, \\ g_{j+1,2k+1} &= p_{j,k}^{[3]}(x_{j+1,2k+1}), k > 1, \\ m_{j+1,0} &= m_{j,0}. \end{aligned} \tag{2.24}$$

We wish to generalise the notion of basis functions, which was introduced in the first chapter, to our new cubic scheme, and a suggestion is to look at interpolation data given on the following two forms

$$f_0' = 1, \qquad f_i = 0,\ \forall\ i \in \{0, \ldots, N\}, \tag{2.25}$$

$$f_k = 1, \qquad f_i = 0,\ \forall i \neq k,\ k \in \{0, \ldots, N\}. \tag{2.26}$$

Let $\psi(x)$ denote the limit function to the data on the form of (2.25), and further let $\phi_k(x)$ denote the limit function to the data on the form of (2.26). In the next chapter we will see that these functions are continuously differentiable by proving that the scheme converge to a continuously differentiable function for any initial data.

Proposition 2.1. $\mathbb{S} = \{\psi(x), \phi_0(x), \ldots, \phi_N(x)\}$ *form a basis for the solution space of the subdivision problem (2.23).*

Proof. We must prove that the functions are linearly independent, and that any solution to the problem can be written as a linear combination of these. Suppose $p \in C^1[x_0, x_n]$.

$$p(x) = b_0\psi(x) + \sum_{i=0}^{n} c_i\phi_i(x) \ \equiv\ 0,$$

so $b_0 = 0$. Assume, to get a contradiction, that $c_i \neq 0$ for some i. Then $c_i\phi(x_i) = c_i \neq 0 \Rightarrow p \neq 0$. Secondly, assume $b_0 \neq 0$ and $p'(x) \equiv 0$ then $p'(x_0) = b_0 \neq 0 \Rightarrow p' \neq 0$. Further it is trivial to see that if

$$g(x) = f_0'\psi(x) + \sum_{i=0}^{n} f_i\phi_i(x), \tag{2.27}$$

Then $g(x_i) = f_i$ for $i \in \{0, 1, \ldots, n\}$ and $g'(x_0) = f_0'$ by definition. $\square$

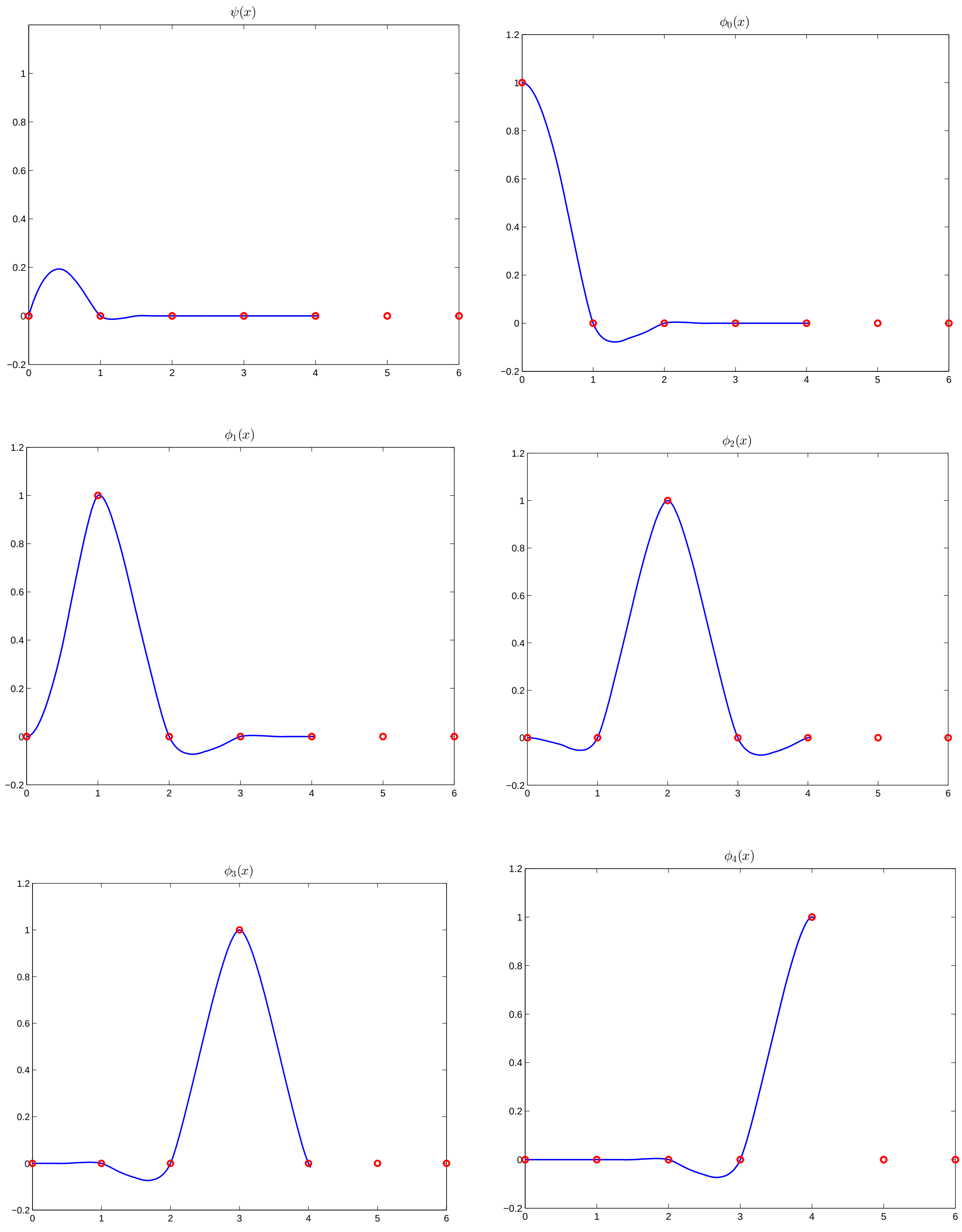

Figure 2.3: Basis functions for *the cubic case*

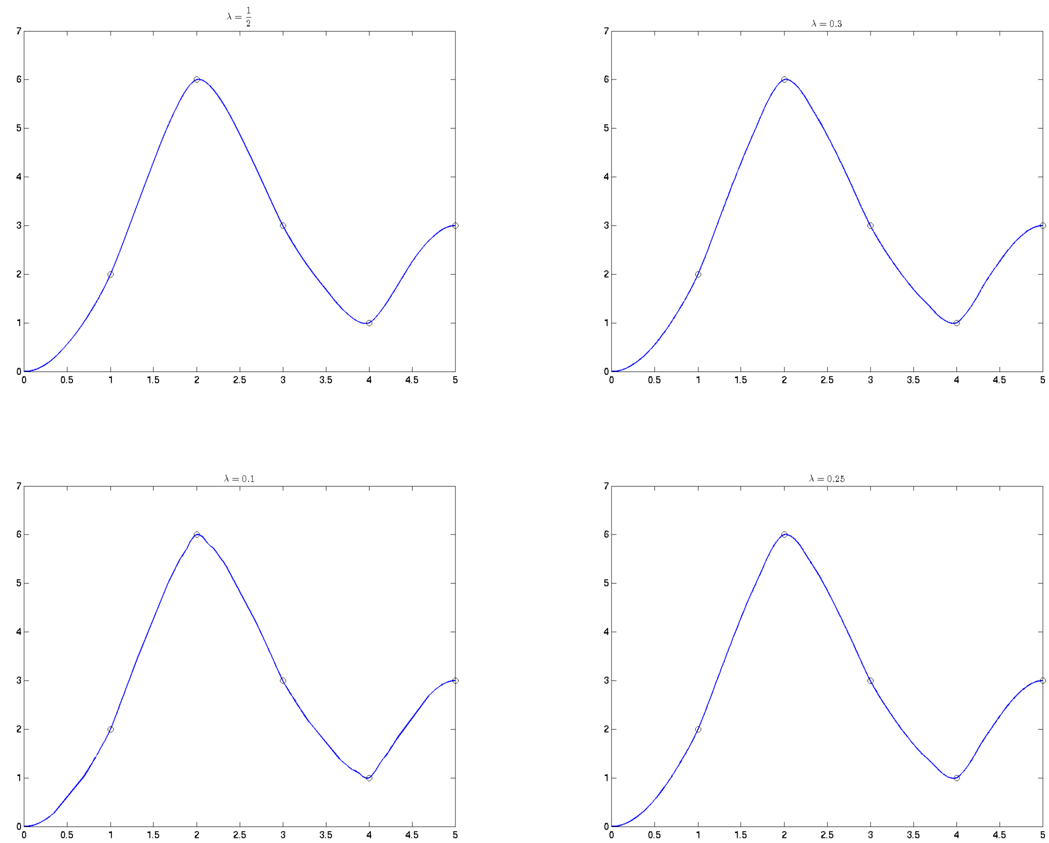

Figure 2.4: Illustration of the different choices of λ

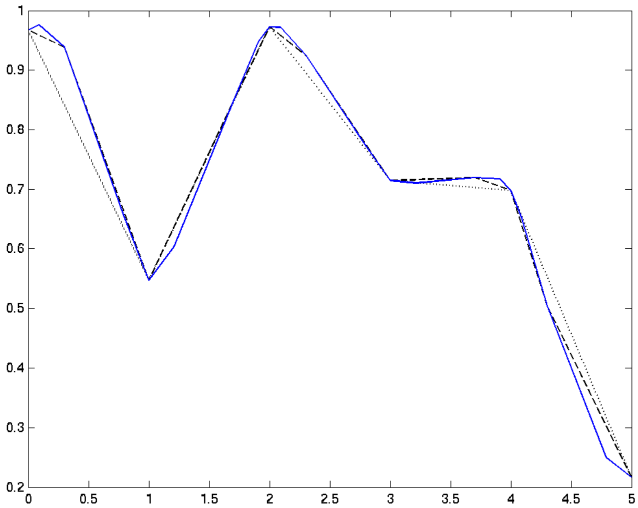

Figure 2.5: The three first iterations

For 7 initial control points the first 6 basic limit function takes the form of Fig 2.3. We implemented the scheme with the option to calculate the limit function for dyadically balanced and homogeneous grids, where the the new grid points were selected by the following rule, given in [4],

$$\begin{aligned} x_{j+1,4k+1} &= (1-\lambda)x_{j,2k} + \lambda x_{j,2k+1} \\ x_{j+1,4k+3} &= \lambda x_{j,2k+1} + (1-\lambda)x_{j,2k+2} \end{aligned} \tag{2.28}$$

here $\lambda \in [0.5, 1)$. This refinement rule of the parameters is such that the grid is homogeneous with $\gamma = \frac{1-\lambda}{\lambda}$, since

$$\gamma \leq \frac{\max\{(1-\lambda)h_{j,2k}, (1-\lambda)h_{j,2k-1}\}}{\lambda h_{j,2k}} \tag{2.29}$$

$$= \frac{(1-\lambda)}{\lambda} \geq 1 \tag{2.30}$$

The effect of choosing different values for λ are shown in figure 2.4. The four curves are generated with values that are equally spaced, but where λ varies. In the upper two figures, the limit function looks quite smooth, while in the middle two the curves look less smooth. Looking at the illustrations we might predict that is exist a premissible range for λ where the limit function is smoother then for other values.

Prescribed endpoint derivatives on both sides

We have also implemented a scheme where we are provided with a derivative at x_n, where use the same logic as for calculating the first odd point to find the last odd point.

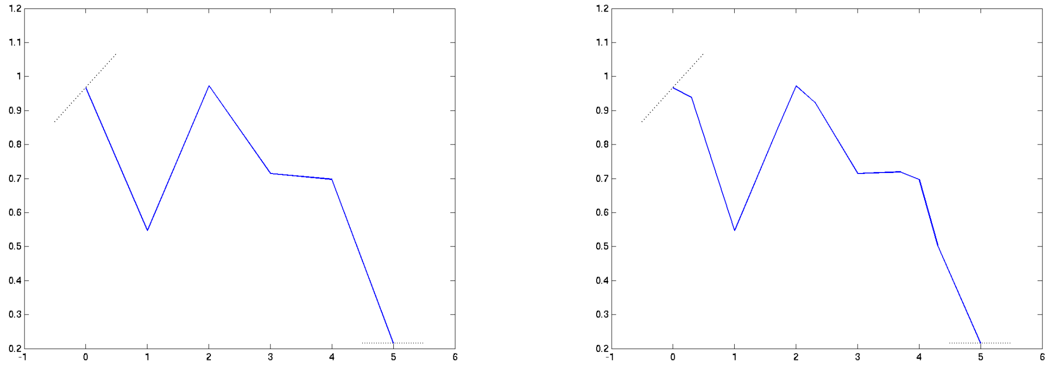

Figure 2.6: Illustration of the cubic scheme with prescribed end point derivatives and x_0 and x_n, iterations $0, 1$.

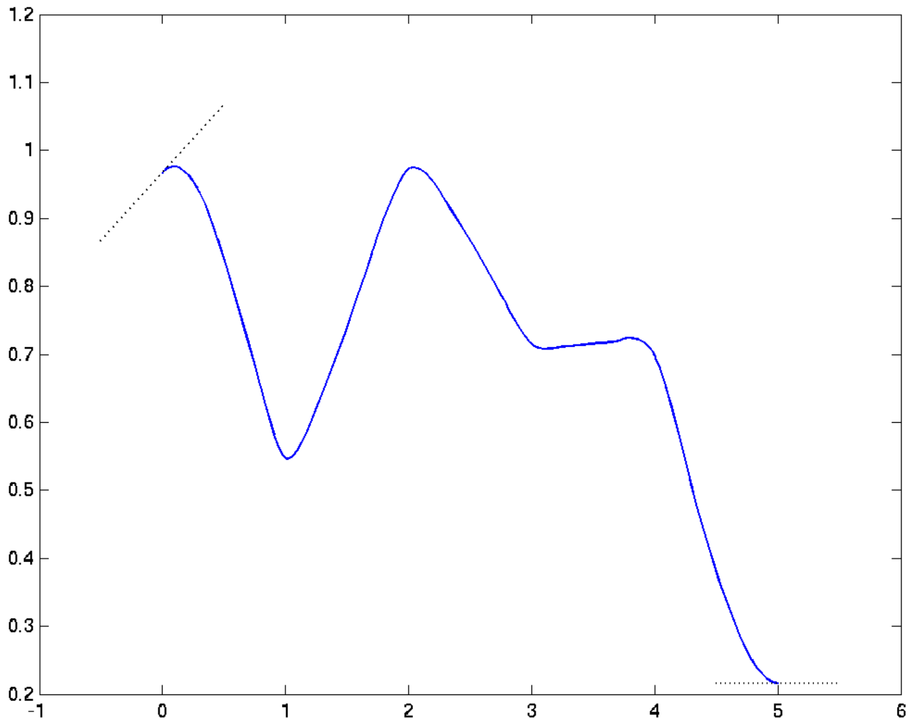

Figure 2.7: Limit curve for the data set shown in 2.6.

Smoothness analysis is not done for this adaption, we predict that it will be similar due to symmetry and we state that this scheme also is C^1. An advantage of this approach is that the limit function interpolates all the given values $f_i, i \in \{0, 1, \ldots, N\}$. With $n_0 = N$, and $n_{j+1} = 2n_j$, the scheme takes the following form

$$\begin{aligned} g_{j+1,0} &= g_{j,0} \\ g_{j+1,1} &= p^{[3]}_{j,0}(x_{j+1,1}), \\ g_{j+1,2k} &= g_{j,k}, 2 < k \leq n_j - 1 \\ g_{j+1,2k+1} &= p^{3}_{j,k}(x_{j+1,2k+1}), 2 < k \leq n_j - 2 \\ g_{j+1,n_{j+1}-1} &= p^{[3]}_{j,n_j-2}(x_{j+1,n_{j+1}-1}), \\ g_{j+1,n_{j+1}} &= g_{j,n_j}, \end{aligned} \tag{2.31}$$

where $p^{[3]}_{j,n_j-2}(x)$ is the osculatory interpolant to the data $g_{j,n_j-2}, g_{j,n_j-1}, g_{j,n_j}$ and a prescribed end point derivative m_{j,n_j}. We also ensure that

$$m_{j+1,0} = m_{j,0}, \tag{2.32}$$

$$m_{j,n_{j+1}} = m_{j,n_j} \tag{2.33}$$

for all j. Analogous to the first cubic scheme we introduced, we can find basis functions for this scheme as well. $\mathbb{S}' = \{\psi_0(x), \phi_0(x), \ldots, \phi_N(x), \psi_n(x)\}$. An illustration of this scheme is shown in Fig 2.6.

Joining two curves with C^1 continuity

An application of both of the schemes presented above is how to join two curves together with C^1 continuity provided that they have the same function value and derivatives at point of intersection. Lets use the first scheme we introduced, where we only had a prescribed derivative in the first point. The problem definition is the given as below:

Assume we are given

$$g(x_i) = g_{0,i} \text{ for } i \in \{-n_g - 2, -n_g - 1, -n_g, \ldots, -1, 0\}$$

and the slope at x_0, $g'(x_0) = m$. Likewise are we given

$$f(x_i) = f_{0,i} \text{ for } i \in \{0, 1, \ldots, n_f, n_f + 1, n_f + 2\}$$

and the slope at x_0, $f'(x_0) = m$. Let the limit curves for $\mathbf{g}_0, \mathbf{f}_0$ be g, f respectively, then we define $F : [x_{-n_g}, x_{n_f}] \mapsto \mathbb{R}$ by

$$F(x) = \begin{cases} g(x) \text{ if } x_{-n_g} \leq x \leq x_0 \\ f(x) \text{ if } x_0 \leq x \leq x_{n_f} \end{cases} \tag{2.34}$$

Then $F(x) \in C^1[x_{-n_g}, x_{n_f}]$, since $g, f \in C^1$ and the derivative coincide at x_0. An example of this application is shown in Fig 2.8 where both the function values and the common derivatives are generated by random uniformly distributed numbers.

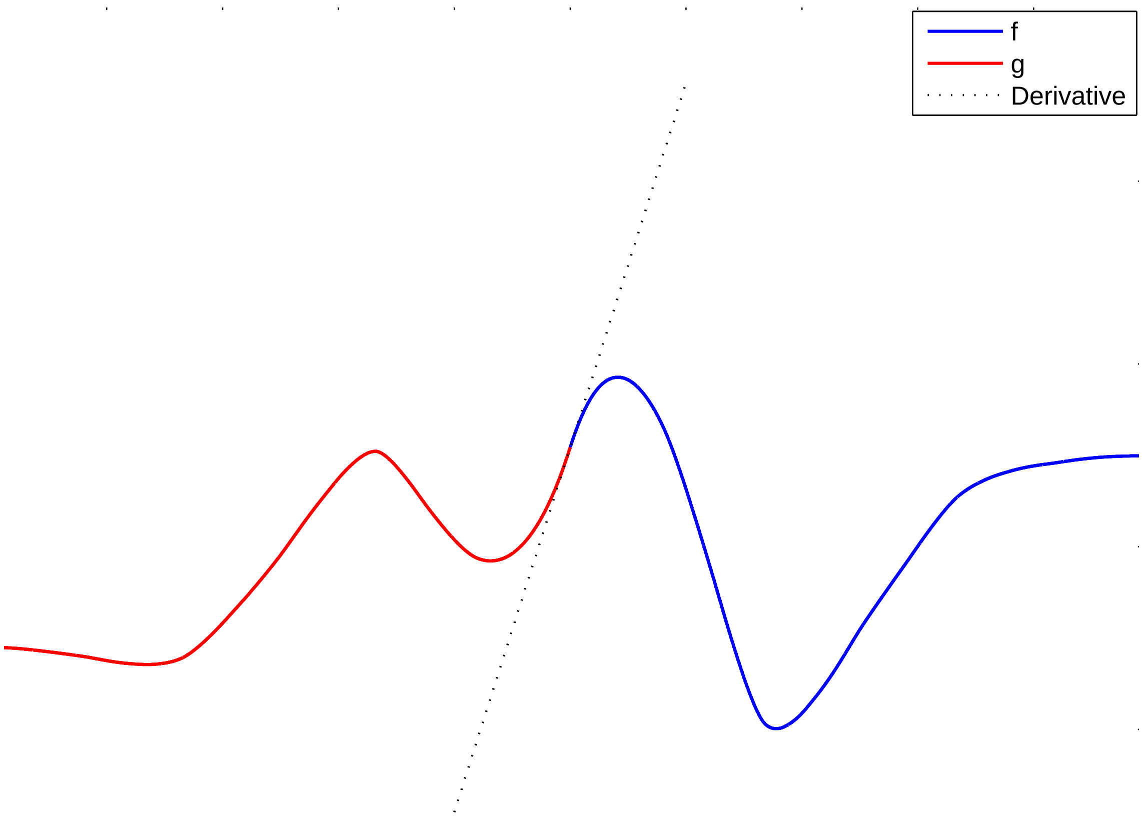

Figure 2.8: Join two curves with C^1 continuity

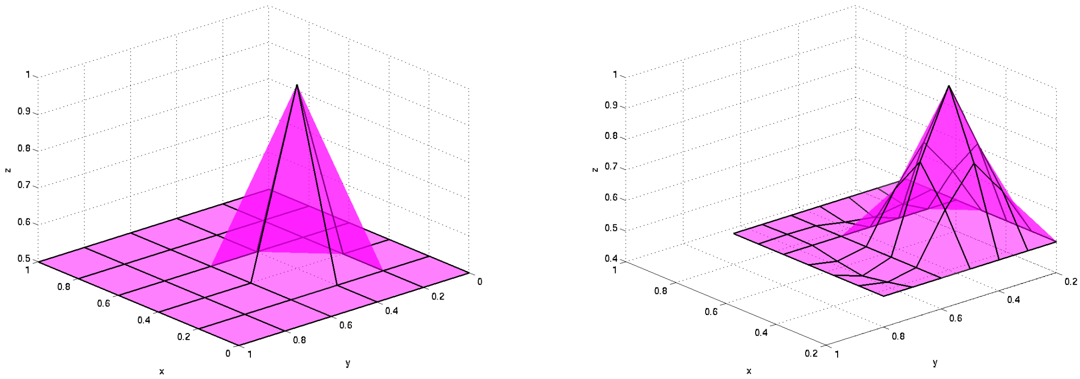

Figure 2.9: Illustration of the four-point tensor product scheme

2.4 A tensor product extension

First we introduce the tensor product four-point scheme based on the four-point scheme. Assume the initial data is given as $\{x_k, y_l, f_{k,l}\}_{k=-2,l=-2}^{n+2,m+2}$, defining a grid in the x, y plane and function values given as heights over this plane. The rectangular grid $\{x_k, y_l\}$ is assumed to be uniformly spaced. We initialise the subdivision process by setting $f_{0,k,j} = f_{k,l}$. For all $j > 0$ we set $f_{j+1,2k,2l} = f_{j,k,l}$. We start the refinement by generating all the *edge points* in x and y direction first by the univariate four-point scheme in both directions

$$f_{j+1,2k+1,2l} = -\frac{1}{16}(f_{j,k-1,l} + f_{j,k+2,l}) + \frac{9}{16}(f_{j,k,l} + f_{j,k+1,l}) \tag{2.35}$$

$$f_{j+1,2k,2l+1} = -\frac{1}{16}(f_{j,k,l-1} + f_{j,k,l+2}) + \frac{9}{16}(f_{j,k,l} + f_{j,k,l+1}) \tag{2.36}$$

Now we are left with all the new face points, which are on the form $f_{j+1,2k+1,2l+1}$. We can find these using the new edge points in the x direction or the y direction. In either case the formula for the new face points takes the form of

$$\begin{aligned} f_{j+1,2k+1,2l+1} = \frac{1}{256}(&f_{j,k-1,l-1} - 9f_{j,k-1,l} - 9f_{j,k-1,l+1} + f_{j,k-1,l+2} \\ &- 9f_{j,k,l-1} + 81f_{j,k,l} + 81f_{j,k,l+1} - 9f_{j,k,l+2} \\ &- 9f_{j,k+1,l-1} + 81f_{j,k+1,l} + 81f_{j,k+1,l+1} - 9f_{j,k+1,l+2} \\ &+ f_{j,k+2,l-1} - 9f_{j,k+2,l} - 9f_{j,k+2,l+1} + f_{j,k+2,l+2}) \end{aligned} \tag{2.37}$$

In this sense, we can say that the scheme is symmetric: the calculation of the new face point using the new edge points in x or y yield the same result.

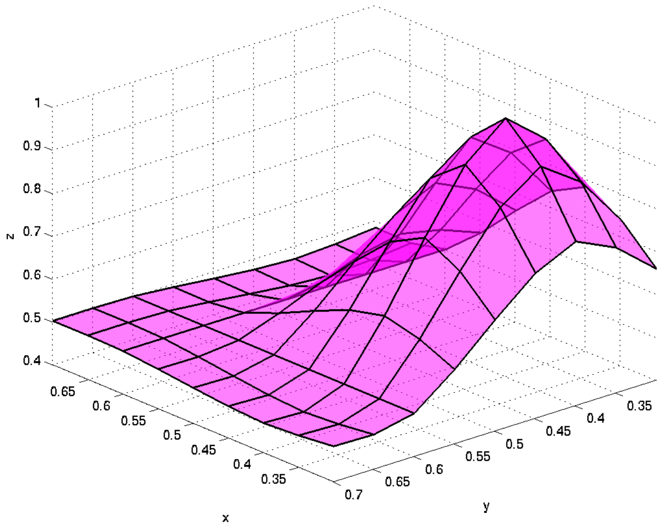

Figure 2.10: Four-point tensor product scheme after 3 iterations

In the first chapter we introduced the basis function $\phi_k(x)$ for the four-point scheme, illustrated in Fig 1.3. Let us define the set of basis functions in x direction

$$\mathbb{S}_1 = \{\phi^0_{-2}(x), \phi^0_{-1}(x), \phi^0_0(x), \phi^0_1(x), \dots, \phi^0_m(x), \phi^0_{m+1}(x), \dots, \phi^0_{m+2}(x)\}$$

and similarly the set of basis function is y

$$\mathbb{S}_1 = \{\phi^1_{-2}(y), \phi^1_{-1}(y), \phi^1_0(y), \phi^1_1(y), \dots, \phi^1_m(y), \phi^1_{m+1}(y), \dots, \phi^1_{m+2}(y)\}$$

Then through a Kroenecker product we can express a the limit function in the tensor product space of $\mathbb{S}_0$ and $\mathbb{S}_1$ as

$$g(x,y) = \sum_{k=-2}^{n+2} \sum_{l=-2}^{m+2} f_{k,l}\phi^0_k(x)\phi^1_l(y) \tag{2.38}$$

With these notions, we will introduce a tensor product scheme based on one of the cubic schemes we defined previously. Recall the cubic scheme introduced in (2.31), where we had prescribed endpoint derivatives at both x_0 and x_n. The initial values where given as $f'(x_0) = m_0, f'(x_n) = m_n$ and $f(x_i) = f_i, i = 0, \dots, n$. We initialised the process by setting $f_{0,i} = f_i \; \forall i$ and $m_{0,0} = m_0, m_{0,n} = m_n$. The subdivision scheme is given as

$$f_{j+1,2k} = f_{j,k}, k = 0, 1, \dots, n_j \tag{2.39}$$

$$f_{j+1,2k+1} = p_{j,k}(x_{j+1,2k+1}), k = 0, 1, \dots, n_j - 1 \tag{2.40}$$

Here we used the normal four-point scheme almost everywhere, except for calculation the first and last odd point. By similar analysis as in the proof of Proposition 2.1 we can describe the limit function of this scheme as a linear combination of basis functions:

$$g(x) = f_0'\psi_0(x) + \sum_{k=0}^{n} f_k\phi_k(x) + f_n'\psi_n(x), \tag{2.41}$$

This gives a total of $n+3$ basis functions. The goal of this section is to generalize this univariate scheme to a bivariate scheme. Assume that we are given $\{f(x_i, y_j) = f_{i,j}\}_{i,j=0}^{n,m}$ over the rectangular grid $\{(x_i, y_j)\}_{i,j=0}^{n,m}$, as well as partial derivatives along the edges of the grid.

$$\begin{aligned}
\frac{\partial f}{\partial x}(x_0, y_l) &= D^{(1,0)} f_{0,l}, \qquad 0 \le l \le m,\\
\frac{\partial f}{\partial y}(x_k, y_0) &= D^{(0,1)} f_{k,0}, \qquad 0 \le k \le n,\\
\frac{\partial f}{\partial x}(x_n, y_l) &= D^{(1,0)} f_{n,l}, \qquad 0 \le l \le m,\\
\frac{\partial f}{\partial y}(x_k, y_m) &= D^{(0,1)} f_{k,m}, \qquad 0 \le k \le n.
\end{aligned} \tag{2.42}$$

An illustration of these initial conditions is given in Fig 2.11, where the blue lines indicates the partial derivatives. We seek a bivariate interpolant $g : [x_0, x_n] \times [y_0, y_m]$ such that

$$\begin{aligned}
g(x_k, y_l) &= f_{k,l} \qquad && 0 \le k \le n, 0 \le l \le m,\\
\frac{\partial g}{\partial x}(x_0, y_l) &= D^{(1,0)} f_{0,l} && 0 \le l \le m,\\
\frac{\partial g}{\partial y}(x_k, y_0) &= D^{(0,1)} f_{k,0} && 0 \le k \le n,\\
\frac{\partial g}{\partial x}(x_n, y_l) &= D^{(1,0)} f_{n,l} && 0 \le l \le m,\\
\frac{\partial g}{\partial y}(x_k, y_m) &= D^{(0,1)} f_{k,m} && 0 \le k \le n.
\end{aligned} \tag{2.43}$$

As we did for the four-point tensor product scheme we define the basis for the solution space in the x-direction and the y-direction.

$$\begin{aligned}
\mathbb{S}_0 &= \{\phi_0^0(x), \phi_1^0(x), \ldots, \phi_n^0(x), \psi_0^0(x), \psi_n^0(x)\}\\
\mathbb{S}_1 &= \{\phi_0^1(y), \phi_1^1(y), \ldots, \phi_m^1(y), \psi_0^1(y), \psi_m^1(y)\}.
\end{aligned} \tag{2.44}$$

The basis for the tensor product space is then given by

$$\mathbb{B} = \mathbb{S}_0 \otimes \mathbb{S}_1 \tag{2.45}$$

Simple counting shows that $\mathbb{B}$ has dimension $(n+3)\times(m+3)$ and we propose that every element $g:[x_0,x_n]\times[y_0,y_m]\mapsto\mathbb{R}$ in this space can be written on the form

$$\begin{aligned} g(x,y) &= \sum_{k=0}^{n}\sum_{j=0}^{m} c_{k,l}\phi_k^0(x)\phi_l^1(y) \\ &+ \sum_{k=0}^{n} d_k\phi_k^0(x)\psi_0^1(y) \\ &+ \sum_{k=0}^{n} e_k\phi_k^0(x)\psi_m^1(y) \\ &+ \sum_{l=0}^{m} h_l\phi_l^1(y)\psi_0^0(x) \\ &+ \sum_{l=0}^{m} v_l\phi_k^1(y)\psi_n^0(x) \\ &+ a_0\psi_0^0(x)\psi_0^1(y) \\ &+ a_1\psi_0^0(x)\psi_m^1(y) \\ &+ a_2\psi_n^0(x)\psi_0^1(y) \\ &+ a_3\psi_n^0(x)\psi_m^1(y). \end{aligned} \tag{2.46}$$

By the expression above and our initial conditions, we are given $(n+3)\times(m+3)-4$ conditions in $(n+3)\times(m+3)$ unknowns showing that the system is underdetermined. Equating and differentiating the expression for the limit surface in (2.46) shows that we must provide the following initial conditions

$$\begin{aligned} c_{k,l} &= f_{k,l}, & 0\le k\le n, 0\le j\le m, \\ d_k &= D^{(1,0)}f_{0,l}, & 0\le l\le m, \\ e_k &= D^{(1,0)}f_{n,l}, & 0\le l\le m, \\ h_l &= D^{(0,1)}f_{k,0}, & 0\le k\le n, \\ v_l &= D^{(0,1)}f_{k,m}, & 0\le k\le n, \\ a_0 &= D^{(1,1)}g(x_0,y_0), \\ a_1 &= D^{(1,1)}g(x_n,y_0), \\ a_2 &= D^{(1,1)}g(x_0,y_m), \\ a_3 &= D^{(1,1)}g(x_n,y_m). \end{aligned} \tag{2.47}$$

We see that we have to supply the mixed derivatives $\frac{\partial^2 f}{\partial x\partial y}(x,y)$ at the four corners of the grid $(x_0,y_0),(x_n,y_0),(x_0,y_m),(x_n,y_m)$. We will also assume that the refinement of the grid is dyadic in both directions.

We tried several different ways of defining the tensor product subdivision scheme, but found it difficult to develop a scheme which was independent of which direction we

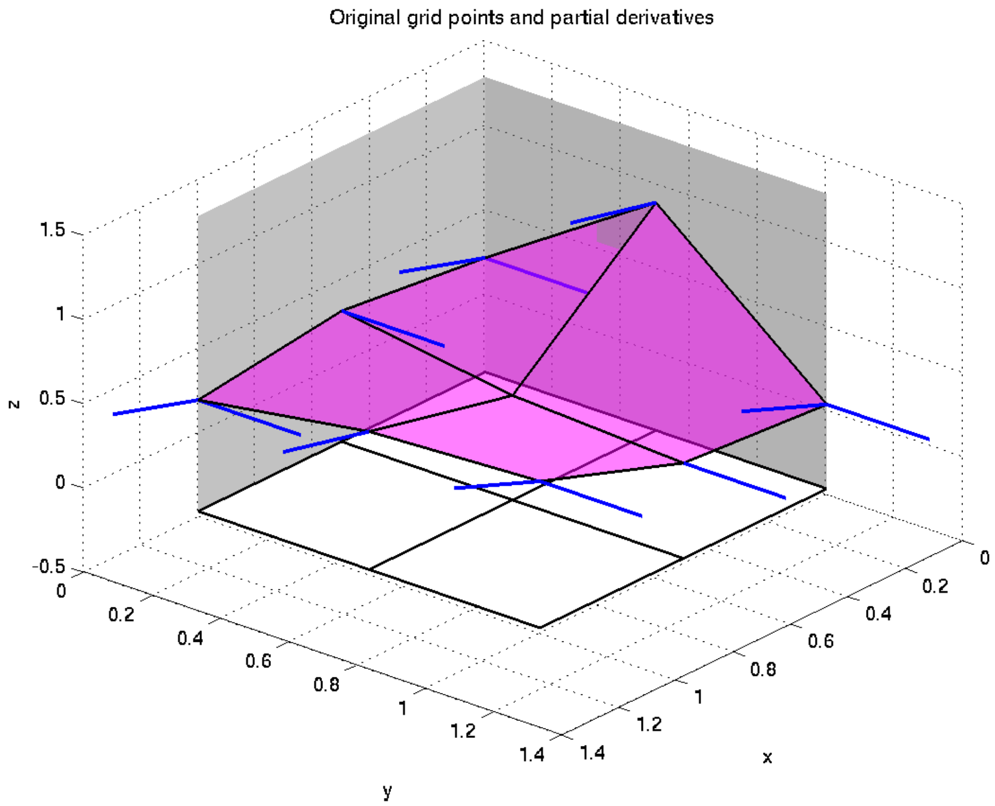

Figure 2.11: Illustration of the initial grid.

would start the refinement. As we discuss for the tensor product four-point scheme, it was symmetric, i.e independent which of the egde points we would refine first. This symmetry is a natural feature, since the refinement should not depend on the orientation of the initial points. It might not we the best suggestion but here we present a numerical scheme for calculating the *discrete* limit function $g : \mathbb{R}^2 \mapsto \mathbb{R}$, the subdivision surface after a finite number of iterations J based on the the *discrete* basis function which we define below. Let the initial values be given in the following manner, in terms of matrices and vectors

$$
\begin{aligned}
G_0 = F &\in \mathbb{R}^{(m+1)\times(n+1)}, & F_{l,k} &= f_{k,l}, k = 0,\dots,n; l = 0,\dots,m \\
\mathbf{h} &\in \mathbb{R}^{1\times(n+1)}, & \mathbf{h}(k) &= \frac{\partial f}{\partial y}(x_k, y_0), k = 0,\dots,n, \\
\mathbf{v} &\in \mathbb{R}^{1\times(n+1)}, & \mathbf{v}(k) &= \frac{\partial f}{\partial y}(x_k, y_m), k = 0,\dots,n, \\
\mathbf{d} &\in \mathbb{R}^{(m+1)\times 1}, & \mathbf{d}(l) &= \frac{\partial f}{\partial x}(x_0, y_l), l = 0,\dots,m, \\
\mathbf{e} &\in \mathbb{R}^{(m+1)\times 1}, & \mathbf{e}(l) &= \frac{\partial f}{\partial x}(x_n, y_l), l = 0,\dots,m, \\
A &\in \mathbb{R}^{2\times 2}, & A(k,l) &= \frac{\partial^2 f}{\partial x \partial y}(x_k, y_l), k = 0, n; l = 0, m.
\end{aligned}
\tag{2.48}
$$

We find the matrices containing the *discrete* basis functions after some number of iterations J. Let $M = m_J, N = n_J$,

$$
\begin{aligned}
\Phi_x &\in \mathbb{R}^{(n+1)\times(N+1)}, & \Phi_x(k,:) &= \{\phi_k^0(x_i)\}, k = 0,\dots,n, i = 0,\dots,N \\
\Phi_y &\in \mathbb{R}^{(M+1)\times(m+1)}, & \Phi_x(:,l) &= \{\phi_l^1(y_i)\}^T, i = 0,\dots,M, l = 0,\dots,m, \\
\Psi_0^x &\in \mathbb{R}^{1\times(N+1)}, & \Psi_0^x(i) &= \psi_0^0(x_i), i = 0,\dots,N, \\
\Psi_n^x &\in \mathbb{R}^{1\times(N+1)}, & \Psi_n^x(i) &= \psi_n^0(x_i), i = 0,\dots,N, \\
\Psi_0^y &\in \mathbb{R}^{(M+1)\times 1}, & \Psi_0^y(i) &= \psi_0^1(y_i), i = 0,\dots,M, \\
\Psi_m^y &\in \mathbb{R}^{(M+1)\times 1}, & \Psi_n^y(i) &= \psi_m^1(y_i), i = 0,\dots,M.
\end{aligned}
\tag{2.49}
$$

We then find $G_J \in \mathbb{R}^{(M+1)\times(N+1)}$, an approximation to the limit surface after J iterations in the following manner.

$$
\begin{aligned}
G_J = {} & \Phi_y G_0 \Phi_x \\
& + \Psi_0^y \mathbf{v} \Phi_x \\
& + \Psi_m^y \mathbf{h} \Phi_x \\
& + \Phi_y \mathbf{d} \Psi_0^x \\
& + \Phi_y \mathbf{e} \Psi_n^x \\
& + A_{0,0} \Psi_0^y \Psi_0^x \\
& + A_{1,0} \Psi_n^y \Psi_0^x \\
& + A_{1,1} \Psi_n^y \Psi_m^x \\
& + A_{0,1} \Psi_0^y \Psi_m^x
\end{aligned}
\tag{2.50}
$$

This scheme will reproduce bicubic functions by construction. An application of this scheme will be how to join two surface with C^1 continuity.

Joining two surfaces

Assume that we generate two surfaces $g : [0,1] \times [0,1] \mapsto \mathbb{R}$ and $\hat{g} : [-1,0] \times [0,1] \mapsto \mathbb{R}$. We assume that $m_g = m_{\hat{g}} = m$ while n_g can differ from $n_{\hat{g}}$. We are joining them along

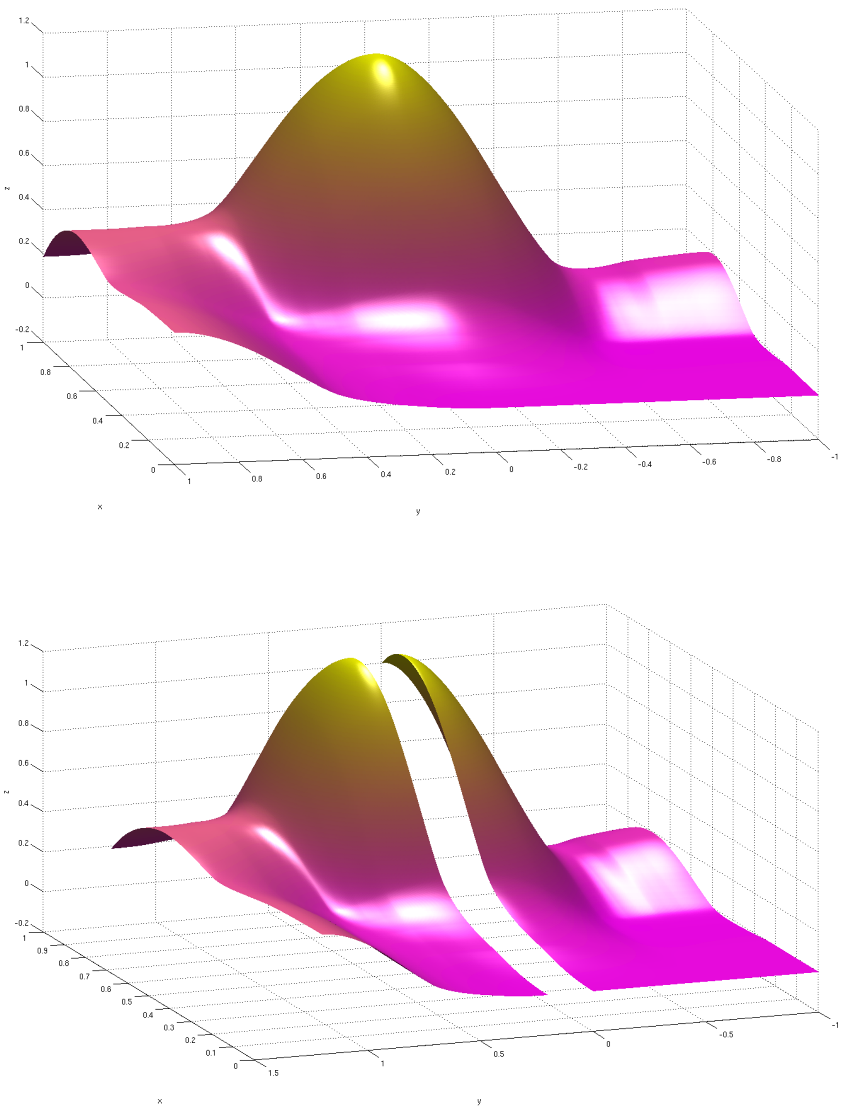

Figure 2.12: Two surfaces joined where the points and partial derivatives on the the common egde are samples from a bicubic function.

the y axis. The initial values for g and $\hat{g}$ are given on the form of (2.47) for $f_{k,l}$ and $\hat{f}_{k,l}$ We hope to ensure C^1 continuity along the common edge $x = 0, y = [0, 1]$, so we assume that we are given initial values on the form

$$
\begin{aligned}
f_{0,l} &= \hat{f}_{n_{\hat{g}},l} && 0 \leq l \leq m, \\
D^{(1,0)} f_{0,l} &= D^{(1,0)} \hat{f}_{n_{\hat{g}},l} && 0 \leq l \leq m, \\
D^{(0,1)} f_{0,0} &= D^{(0,1)} \hat{f}_{n_{\hat{g}},0}, \\
D^{(0,1)} f_{0,m} &= D^{(0,1)} \hat{f}_{n_{\hat{g}},m}, \\
D^{(1,1)} f_{0,0} &= D^{(1,1)} \hat{f}_{n_{\hat{g}},0}, \\
D^{(1,1)} f_{0,m} &= D^{(1,1)} \hat{f}_{n_{\hat{g}},m}.
\end{aligned}
\tag{2.51}
$$

Visual appearance

The initial condition of the left part of the surface in figure 2.13 are given as in figure 2.15. The function values are 1 in the first two rows and 0.5 on the last one. The partial and mixed derivatives are all set to zero. Similarly for the right part the surface, the initial values are given as 0.5 in the first row and 0 on the two last rows. As we can see from Fig 2.13, the join does not appear to be very smooth, but we know the that two surfaces are not C^2, only C^1, since the univariate scheme is not C^2. If we plot the surface normals see that the normal vector changes abruptly on the line of intersection. As light is calculated using the surface normals this can be the reason that the shading and reflection is different on this line. Based on the cross section plot we can predict that this reflection phenomenon also can be due to the change in curvature in this point. There is no direct discontinuity in the derivative of the individual curves in the image, based the cross section plot. We did not really have the time to go into the depth on this problem, but the scheme produce more pleasing results for other data sets, as shown in Fig 2.14. We noticed that especially when the initial values on the common egde where given as samples from a bicubic function the result was better and the line of intersection was practically invisible, this is illustrated in Fig 2.12.

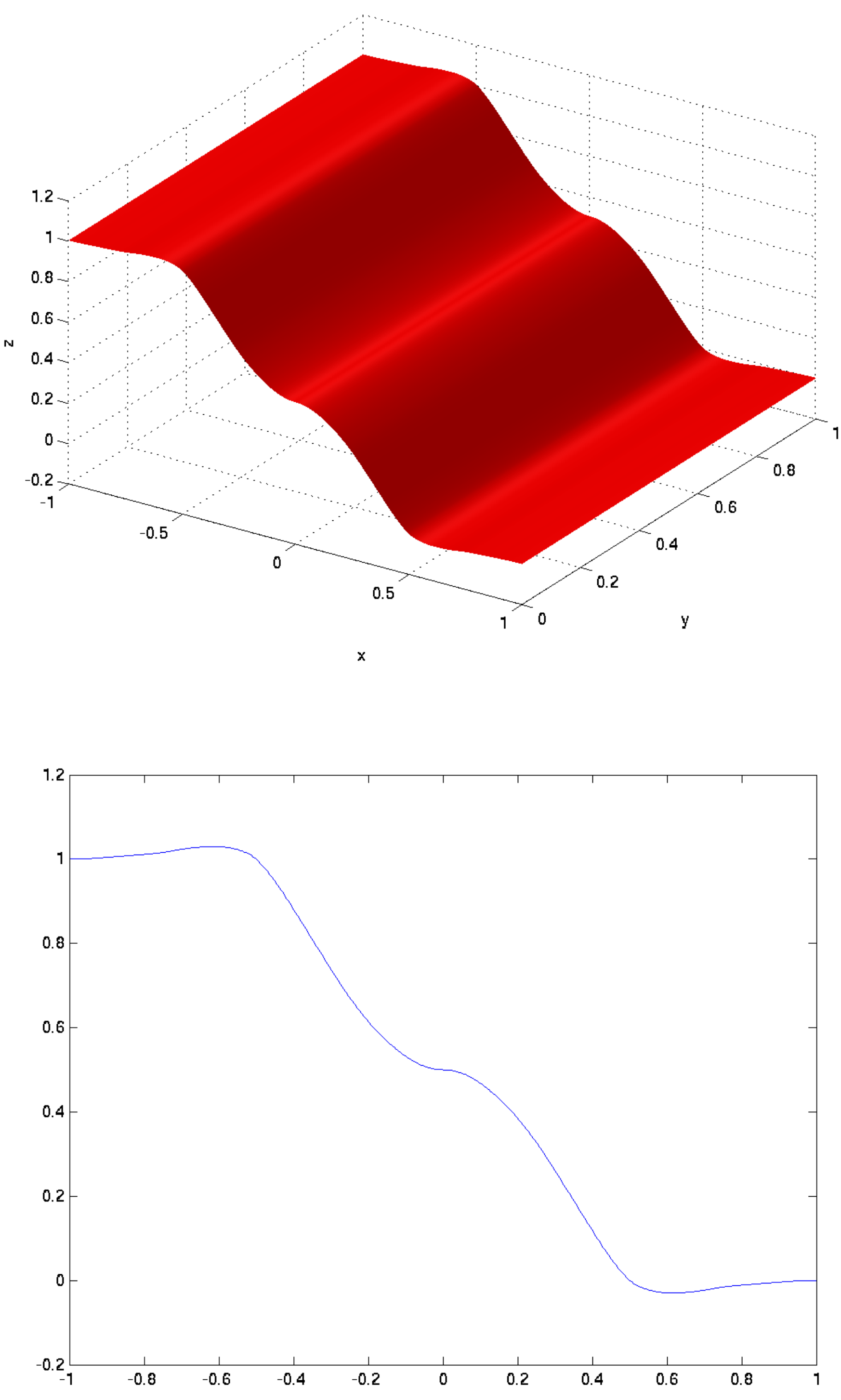

Figure 2.13: First example of joining two surfaces that share a boundary, the second image shows a cross section curve of the surface.

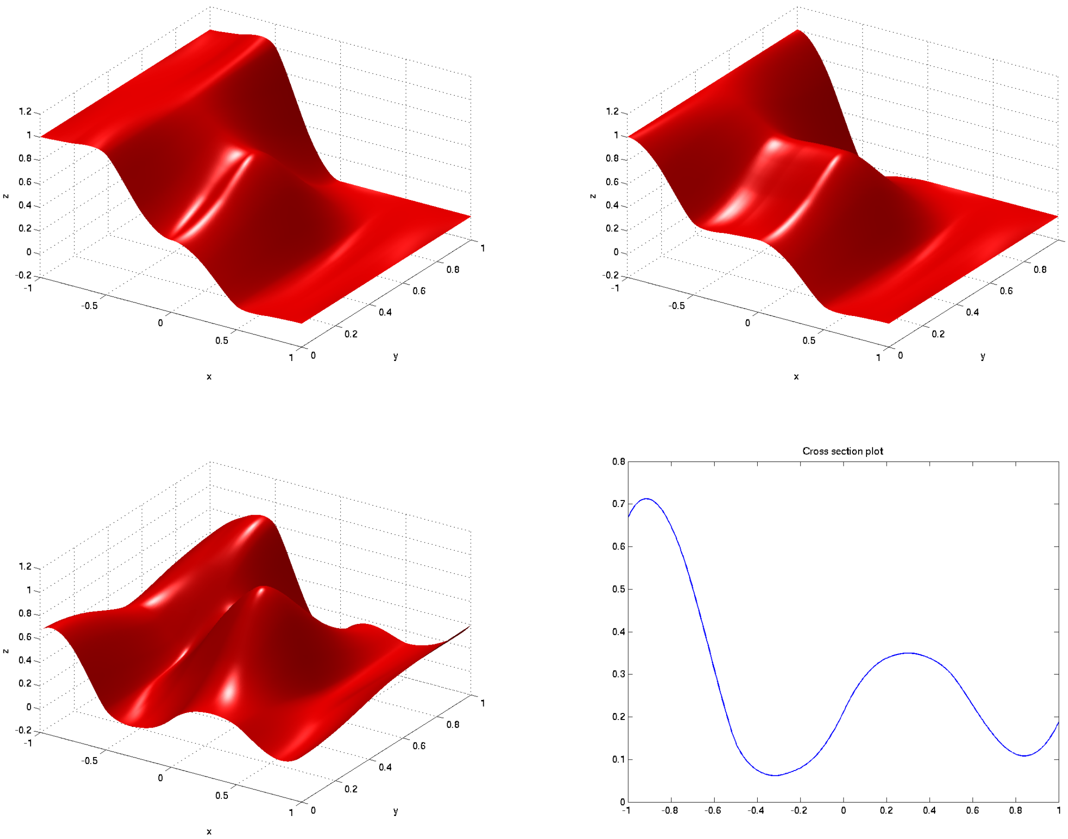

Figure 2.14: Some additional example of surfaces joined

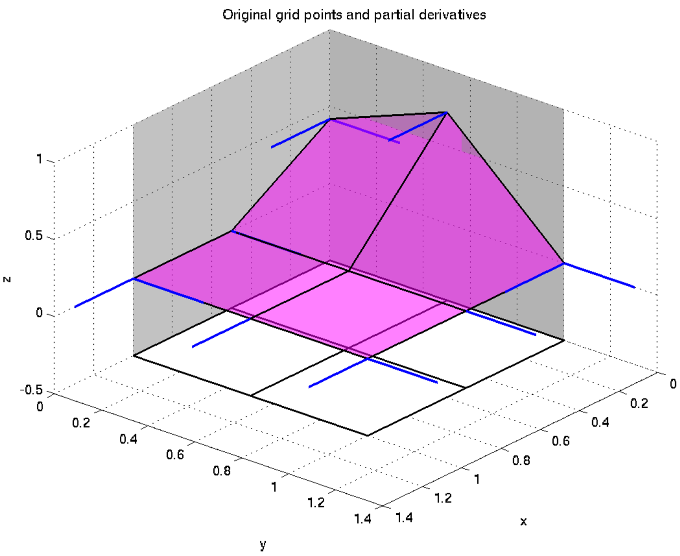

Figure 2.15: Illustration of initial conditions for the surface in Fig 2.13

2.5 Combined subdivision schemes

In the search for related material we came across the Ph.D-thesis of Adi Levin titled *Combined Subdivison Schemes* submitted March 2000. Combined subdivison schemes take into account prescribed boundary conditions of a curve or a surface. The boundary conditions can be stated as cross derivatives of a given function or a curve on the boundary which should be interpolated by the subdivision surface. The main idea for a combined subdivision scheme is to alter the subdivison rules near the boundary of the object, and keep the ordinary rules in the interior. This resembles the way we chose to define our new schemes. The need for a prescribed boundary of a geometric object have applications in modelling of mechanical parts for example. A mechanical part often consist of several different components and each of them being a smooth surface. We can design the part as a union of smooth surfaces that share boundaries. To model these creases we often need more information than we do when modelling the smooth interior of the object. When justifing the need for combined subdivison scheme Levin argues that traditonally when modelling the boundary or the intersection between two geometric objects we will run into some implementational and mathematical problems. First of all, boundaries of an object is in several applications, an important feature, both for the visual accuracy and for further computations, and if two surfaces are to meet in some plane, it is crucial that there is no gaps between them where the intersect. He further points out that even for tensor product surfaces of degree three, the intersection between two such surfaces may be a polynomial of high degree therefore is not very suitable for processing the surfaces on a computer. In turn, we might want to approximate the curve within an error tolerance, leading to a a surface with actual gap between two of its parts. When using a combined subdivison scheme, the prescibed interpolatory condition of the boundary curve is included in the contruction of the whole object. Levin describes the recipe for this as calculating the boundary and requiring that both surfaces interpolate the curve. Another application of combined subdivision scheme to join two surfaces together in a smooth fashion. As we know by now, the subdivison process can be viewed as a linear process, related to the uniform subdivison operator S, which we have discussed in terms of bi-infinite matrices.

$$p_{j+1} = Sp_j$$

Combined subdivison schemes can be view as

$$p_{j+1} = Sp_j + \text{ (boundary contributions)}$$

We start as usual with our *control points*, but we are also provided additional information to calculate the surface near a certain given hyperplane, given as either a given boundary curve f and possibly partial derivatives of this curve of some order k. Both given as continuous values of a given function f. He defines the *combined subdivison operator* as an operator on both discrete control points and a functional operating on continuous information from this given function. The author defines that the subdivison scheme such that it acts uniformly away from a closed subset ε in $\mathbb{R}^s$ that describes the boundary. This subset is called the *exterior* of the scheme. The *combined subdivison operator* is an operator

$$B : l(\mathbb{Z}^s) \times C^k(\mathbb{R}^s) \mapsto l(\mathbb{Z}^s)$$

where $l(\mathbb{Z}^s)$ is the space of all sequences from $\mathbb{Z}^s \mapsto \mathbb{R}$. The set of control points belong to $l(\mathbb{Z}^s)$ and $C^k(\mathbb{R}^s)$ is the space of k times differentiable functions over $\mathbb{R}^s$. The operator is defined such that

$$B(p, f)(\alpha) = Sp(\alpha), \qquad \forall \alpha \in \mathbb{Z}^s \setminus (\varepsilon + \Omega), \forall p \in l(\mathbb{Z}^s), \qquad f \in C^k(\mathbb{R}^s)$$

where k is the order of derivatives of f that must be continuous for $B(p, f)$ to be well-defined, and will denonte the order of the scheme. Ω is the support of the uniform subdivison operator S and is also the support of B. A combined subdivision scheme B of order k is based on S with support Ω, whose exterior is ε.

The work of Levin in [14] is extensive and the topics go way beyond the scope of this thesis, but most are quite relevant for us and very interesting. To get a grasp on the subject however, we choose to work through on example given in Chapter 4 of **??**.

The tensor product four-point scheme interpolating boundary curve on ε

Here $\varepsilon = \{x \in \mathbb{R}^2 \mid x_1 = 0\}$, and the task is to interpolate a given curve on this plane. Throughout we assume that the grid is refined regularly in both x and y direction. Let p we the given values, defined over a subset in $\mathbb{Z}^2$. The subdivision scheme B works as the regular tensor product four-point scheme everywhere, except on ε where it uses samples of the given function

$$B(p, f)(\alpha) = \begin{cases} f(\alpha), & \alpha \in \mathbb{Z}^2 \bigcap \varepsilon, \\ Sp(\alpha), & \alpha \in \mathbb{Z}^2 \setminus \varepsilon. \end{cases} \tag{2.52}$$

The subdivison scheme can be described by the following process

$$\begin{aligned} p^0 &= p \in l(\mathbb{Z}^s), \\ p^{n+1} &= B(p^n, f(2^{-n}\cdot)), \qquad n = 0, 1, \ldots \end{aligned} \tag{2.53}$$

So the scheme is a stationary process with a single operator, thus has a similar structure as in the standard setting. The boundary conditions are expressed through the function f and are defined over ε. The limit function of this process is defined as follows

Definition 2.3 ([14] Definition 3.1.5)**.** *We say that $F \in C(\mathbb{R}^s \setminus \varepsilon)$ is the limit function of the combined scheme (2.53), if for every $x \in \mathbb{R}^s$ ε there exists an open domain $D_x \subset \mathbb{R}^s \setminus \varepsilon, x \in D_x$, such that*

$$\lim_{n\to\infty} \|p^n - F(2^{-n}\cdot)\|_{\infty, \mathbb{Z}^s \cap (2^n D_x)} = 0. \tag{2.54}$$

We denote $B^\infty(p, f) = F$. Uniqueness follows from the definition.

From the way the operator is defined we will see that the limit function of B will coincide with the limit function of S *far enough* from the exterior. We say that a uniform subdivision operator belongs to the class C^m if it is uniformly convergent, and for every initial point set p, the limit curve $S^\infty p \in C^m(\mathbb{R}^s)$, where $C^m(\mathbb{R}^s)$ denotes the space of all functions over $\mathbb{R}^s$ which are m-times differentiable. Now that we have defined the limit function, we can state a lemma concerning sufficient conditions for the existence of the limit function.

Lemma 2.1 ([14] Lemma 3.1.6). *If S, our original uniform subdivision operator, is C^m then the limit function $B^\infty(p, f)$ exists for all initial values p and all $f \in C^k(\mathbb{R}^s)$. Futhermore, $B^\infty(p, f) \in C^m(R^s \backslash \varepsilon)$ and*

$$B^\infty(p, f)(x) = S^\infty p^r(2^r x),$$

whenever $x \in \mathbb{R} \backslash (\varepsilon + 2^{-r}\Omega)$ and $r \in \mathbb{Z}_+$

Note that now we have only defined the limit function in the interior, we wish to extend the limit function to be valid for the entire index domain, also the boundary.

Definition 2.4 ([14] Extended limit function).

$$B_e^\infty(p, f) = \begin{cases} B^\infty(p, f)(x), & x \in R^s \backslash \varepsilon \\ f(x), & x \in \varepsilon \end{cases} \tag{2.55}$$

From this it is clear that given B, we want to investigate the smoothness of the limit function near the boundary, and how smoothly if connects to f as we approach $\partial\varepsilon$. This is analougous to what we will do in the analysis for *the cubic case.* E.i we know the smoothness of the ordinary four-point scheme, and want to investigate how the limit function behaves in the area affected by the change. He defines some classes of smoothness which will be used to define the smoothness of our limit function.

Definition 2.5 ([14]The class C^m_+). *$B \in C^m_+$ if $B^\infty(p, f)$ can be extended to a function in $C^m(\overline{\mathbb{R}^s \backslash \varepsilon})$ for any set of initial data p and every smooth enough f.*

Definition 2.6 ([14]). *We say that $B \in C^m$ if $B \in C^m_+$ and*

$$D^j B^\infty(p, f)|_{\partial\varepsilon} = D^j f|_{\partial\varepsilon} \qquad \forall j \in \mathbb{Z}^s, |j| \le m \tag{2.56}$$

for every initial data p and smooth enough f.

When $B \in C^m$ it generates limit functions that are not only smooth, but have a C^m connection to the given function f. The order of this last class is determined by how many derivatives of the limit function and f that agreee at the boundary of the exterior. So the smoothness analysis will establish m_1,m_2 as high as possible such that

$B \in C^{m_1} \bigcap C_+^{m_2}$ where $m_1 \leq m_2$. We have assumed as for the stationary case that the operator B is linear, then so is the limit function, thus we can write

$$B^\infty(p, f) = B^\infty(p, 0) + B^\infty(0, f)$$

Further, locality gives

$$B^\infty(p, 0) = \sum_{\alpha \in \mathbb{Z}} p(\alpha) B^\infty(\delta_\alpha, 0)$$

where $B^\infty(\delta_\alpha, 0)$ are the homogeneous basis functions of B. From this it is clear that the smoothness of the limit function can be deduced from these basis functions and form the application of B to the zero data set, $B^\infty(0, f)$. There are only a finite number of basis function since the support of B is finite. It is a well known fact that some uniform subdivision schemes reproduce polynomials up to a certain degree and this fact is used extensivly to determine the smoothness of the limit function. We saw this in the section on derived schemes. Also for combined subdivision schemes this reproduction ability is an important and desirable feature both in the derivation and analysis. It is shown that instead of looking at the space of smooth enough functions f to determine the smoothness classes C_+^m and C^m, we can in some cases reduce the problem by looking at functions in some polynomial space Π_r, making the analysis easier. In the example we survey we see that the order of the subdivision scheme B is 0, since we only work with function values of f. The tensor product four-point scheme is known to be C^1. According to [14], sufficient condition for C_+^m and C^m reduces to consider f in the polynomial case and it suffice to consider polynomials of degree 1. The analysis of this example done in [14] shows that the limit function $B^\infty(p, f) \in C^0 \bigcap C_+^1$. C_+^1 implies that the operator B generates limit functions that can be extended to C^1 functions on the entire plane. It is C^0 by construction since it reproduce linear polynomials, while it is *not* C^1; consider $f(x) = x_1$, and the zero data as initial data p, them $B^\infty(p, f) = 0$ since $f|_\varepsilon = 0$. But $D^{(1,0)} f|_\varepsilon = 1$, while $D^{(1,0)} B^\infty(p, f)|_\varepsilon = 0$. The conclusion is that if $f \in C^1$, $B^\infty(p, f)$, then is C^1 and interpolates the given function f on ε. We implemented the method described in this section for $f(x) = sin(x)$, see the figures (2.16),(2.17).

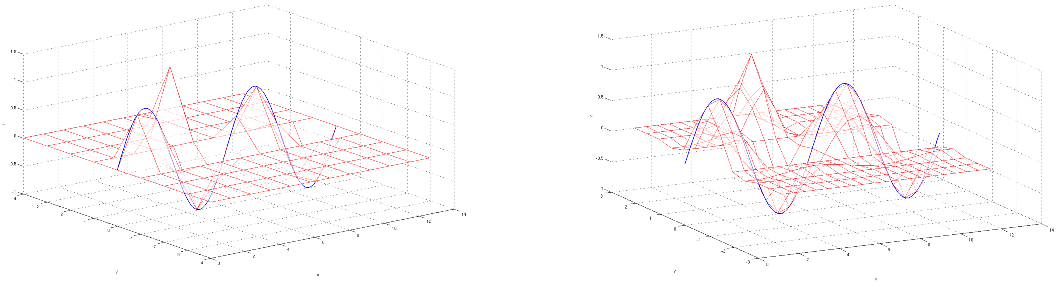

Figure 2.16: Initial and first refinement grid using a combined subdivision scheme

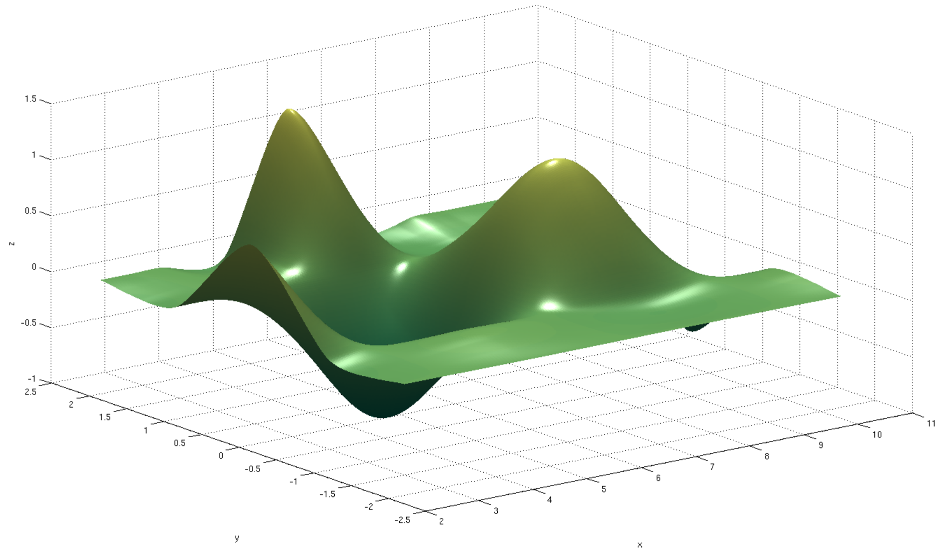

Figure 2.17: Limit surface of a combined subdivision scheme

2.6 A modified four-point scheme

In our first example of a new subdivision scheme, we provide a first order derivative at the first point, reducing the conditions from four to three. The purpose of this adjustment is to add control of the derivative at the beginning of our curve, so we also seek to prove that the derivative of the limit curve exists, is continuous and interpolates the given derivative value. This adjustment could be done similarly at the end, as stated previously. Another way to describe a modification of the four-point scheme was given by Cai Zhijie [26]. Here it is suggested a way of finding an expression for the derivative based on a finite set of the given control point. The method of joining two curves with C^1 continuity will then require some restriction on the control points on each of the curves at every refinement level. We briefly review results concerning a modification of stationary four-point scheme with tension parameter $\omega = \frac{1}{16}$ as given in (1.37). The modified four-point scheme is defined as follows.

$$\begin{aligned} p_{j+1,2k} &= p_{j,k},\ 0 \leq k \leq 2^j n, \\ p_{j+1,2k+1} &= \frac{9}{16}(p_{j,k} + p_{j,k+1}) - \frac{1}{16}(p_{j,k-1} + p_{j,k+2}),\ 1 \leq k \leq 2^k - 2, \\ p_{j+1,1} &= \frac{5}{16}p_{j,0} + \frac{15}{16}p_{j,1} - \frac{5}{16}p_{j,2} + \frac{1}{16}p_{j,3}, \\ p_{j+1,2^{k+1}n-1} &= \frac{1}{16}p_{j,2^jn-3} - \frac{5}{16}p_{j,2^jn-2} + \frac{15}{16}p_{j,2^jn-1} + \frac{5}{16}p_{j,2^kn}, \end{aligned} \tag{2.57}$$

Here the first odd point $p_{j+1,1}$ is chosen such that $(p_{j+1,0}, p_{j+1,1}, p_{j+1,2}, p_{j+1,3}, p_{j+1,4})$ are on the same cubic curve. The same approach is used for the last odd point. As for the stationary ordinary four point scheme, this scheme has cubic precision. The first and second theorem of this article states that the scheme converge to a continuously differentiable function $f : [0, n] \mapsto \mathbb{R}$. Concerning the derivatives at the end points, the author finds an explicit formula for it.

$$f'(0) = \frac{2^{j+3}}{6}[\frac{1}{4}p_{j,3} - \frac{9}{8}p_{j,2} + \frac{18}{8}p_{j,1} - \frac{1}{2}p_{j,0}] \tag{2.58}$$

The last theorem of this article states under what condition we can join two C^1 curves with C^1 continuity using the modified four-point scheme.

Theorem 2.1 ([26] 4.2)**.** *Given two sequences of initial data $\{p_i\}$ and $\{q_i\}$ with assosiated limit functions f and g respectively and suppose $f(0) = p_0$ and $g(0) = q_0$. Let*

$$F(t) = \begin{cases} f(-t) & t \leq 0, \\ g(t) & t \geq 0. \end{cases}$$

Then

- $F \in C^0 \Leftrightarrow p_0 = q_0$

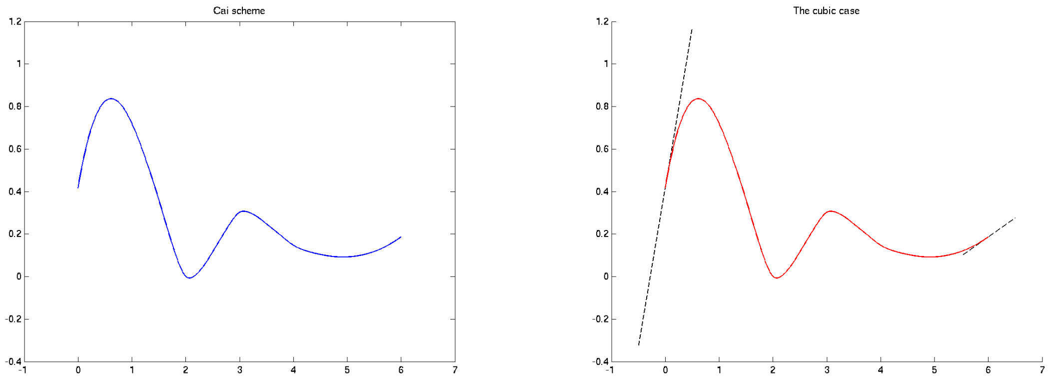

Figure 2.18: Comparison of the scheme in (2.57) and *the cubic case*

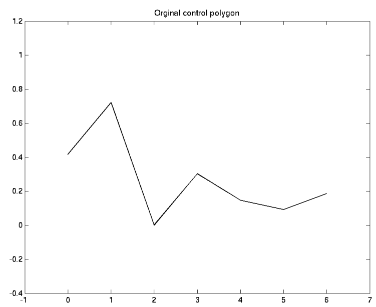

Figure 2.19: Initial control polygon as input to the scheme in (2.57)

- $F \in C^1 \Leftrightarrow p_0 = q_0$ *and*

$$\frac{1}{4}[p_{j,3} - \frac{9}{8}p_{j,2} + \frac{18}{9}p_{j,1} - \frac{11}{16}p_{j,0}] \quad = -[\frac{1}{4}q_{j,3} - \frac{9}{8}q_{j,2} + \frac{18}{9}q_{j,1} - \frac{11}{16}q_{j,0}] \tag{2.59}$$

From the C^1 condition above we see that this method impose a relation between eight points. Four of the control points of g and four of f, and even between these control points at each refinement level. We were not sure how to implement this relationship, but since the first derivative of the limit function is specified (2.58) we tried to compare our *cubic case* to this scheme by the means of an example. The derivatives given (2.58) were input to our cubic scheme, as well as the same function values. The output of the schemes actually coincide, which verifies the validity of the scheme we have introduced. By validity we mean the fact that the prescribed end derivative is interpolated.

3 Analysis of the cubic case

3.1 Introduction

In the following chapter we hope to prove that the subdivision scheme introduced in the previous chapter, *the cubic case* where $s = 1, d = 3$, converge to a continuous limit function with a continuous first derivative, and further argue that the limit curve has the precribed derivative value at x_0. Let us first recall the definition of the problem. Given a finite sequence of function values

$$\{f_i\}_{i=0}^N,$$

associated with a stricly increasing sequence of parameters

$$\{x_i\}_{i=0}^N,$$

and a value for the slope at x_0,

$$f'(x_0) = m_0.$$

We seek a smooth interpolant $g : [x_0, x_n] \mapsto \mathbb{R}$, where $n = N - 2$, such that

$$\begin{aligned} g(x_k) &= f_k, \ k = 0, \dots, n, \\ g'(x_0) &= m_0. \end{aligned} \tag{3.1}$$

We initialize the subdivision process by setting

$$g_{0,i} = f_i, 0 \leq i \leq N.$$

The scheme takes the following form

$$\begin{aligned} g_{j+1,2k} &= g_{j,k}, 0 \leq k \leq n_j, \\ g_{j+1,2k+1} &= p_{j,k}^{[3]}(x_{j+1,2k+1}), 0 \leq k \leq n_j, \\ m_{j+1,0} &= m_{j,0}, \end{aligned} \tag{3.2}$$

where the cubic interpolants $p_{j,k}^{[3]}(x)$ are defined in the previous chapter and

$$n_0 = N, n_j = 2n_{j-1} - 2.$$

In the following chapter, the notation and definitions regarding divided differences and Newton's interpolation formula will be used extensively. We investigate the smoothness of the interpolant g under the assumption that the multi-level grid $\mathbf{X}$ is dyadically balanced. The general approach throughout this chapter is based on a paper by Floater [11], *A piecewise polynomial approach to analyzing interpolatory subdivision.*

3.2 Convergence and differentiability

Here we discuss the notion of convergence and differentiability of the limit function of *the cubic case*. We take our domain to be $[x_0, x_n] = [0, n]$ as all other cases can be dealt with through scaling and translation. Recall that $n = N-2$, and let j denote the current refinement level. Let $I_{j,k} := [x_{j,k}, x_{j,k+1}]$ and let $s_j : [0, n] \mapsto \mathbb{R}$ denote the piecewise cubic function

$$s_j(x) = p_{j,k}^{[3]}(x), \quad x \in I_{j,k}, \tag{3.3}$$

where $p_{j,k}^{[3]}(x)$ are the cubic interpolants defined as in the previous chapter, but lets restate them here. For simplicity, we denote the nodal polynomials on level j by

$$\psi_{j,k}^{[i]}(x) := (x - x_{j,k})(x - x_{j,k+1}) \dots (x - x_{j,k+i}). \tag{3.4}$$

Then the cubic interpolants can we expressed as

$$p_{j,0}^{[3]}(x) = [0]g_{j,0} + \psi_{j,0}^{[0]}[0,0]g_{j,0} + (\psi_{j,0}^{[0]})^2[0,0,1]g_{j,0} + \psi_{j,0}^{[0]}\psi_{j,0}^{[1]}[0,0,1,2]g_{j,0} \tag{3.5}$$

$$p_{j,k}^{[3]}(x) = [-1]g_{j,k} + \psi_{j,k-1}^{[0]}[-1,0]g_{j,k} + \psi_{j,k-1}^{[1]}[-1,0,1]g_{j,k} + \psi_{j,k-1}^{[2]}[-1,0,1,2]g_{j,k} \tag{3.6}$$

We define the limit function g as the limit of the piecewise cubics

$$g(x) = \lim_{j \to \infty} s_j(x). \tag{3.7}$$

Recall the supremum norm for a bounded function $f : \mathbb{R} \mapsto \mathbb{R}$

$$\|f\| = \sup_{x \in \mathbb{R}} |f(x)|. \tag{3.8}$$

First we want to show that the sequence of bounded and continuous functions $\{s_j\}$ form a Cauchy sequence in the supremum norm and hence converge to a continuous function by Lemma 1.1. To prove that the s_j form a Cauchy sequence we will need an expression for the difference between two consecutive functions s_{j+1} and s_j.

Lemma 3.1. *For $j \geq 0$, $k \geq 0$*

$$s_{j+1}(x) - s_j(x) = \begin{cases} 0, & x \in I_{j+1,0}, \\ -\psi_{j+1,2k}^{[2]}(x) h_{j+1,2k+3}, g_{j+1,2k}^{[4]}, & x \in I_{j+1,2k+1}, \\ \psi_{j+1,2k}^{[2]}(x) h_{j+1,2k-2} g_{j+1,2k-2}^{[4]}, & x \in I_{j+1,2k}. \end{cases} \tag{3.9}$$

Proof. Set $p_{j,k}^{[3]}(x) := p_{j,k}(x)$. The case for the even numbered intervals was proved in Lemma 1 of [11]. We will now prove the odd case, and the the special cases where $x \in I_{j+1,0}$ or $x \in I_{j+1,1}$ affected by the new subdivision rule. First, let $x \in I_{j+1,2k+1}$

$$\begin{aligned} s_{j+1}(x) - s_j(x) &= p_{j+1,2k}(x) - p_{j,k-1}(x) \\ &= c_1(x - x_{j+1,2k})(x - x_{j+1,2k+1})(x - x_{j+1,2k+2}), \end{aligned}$$

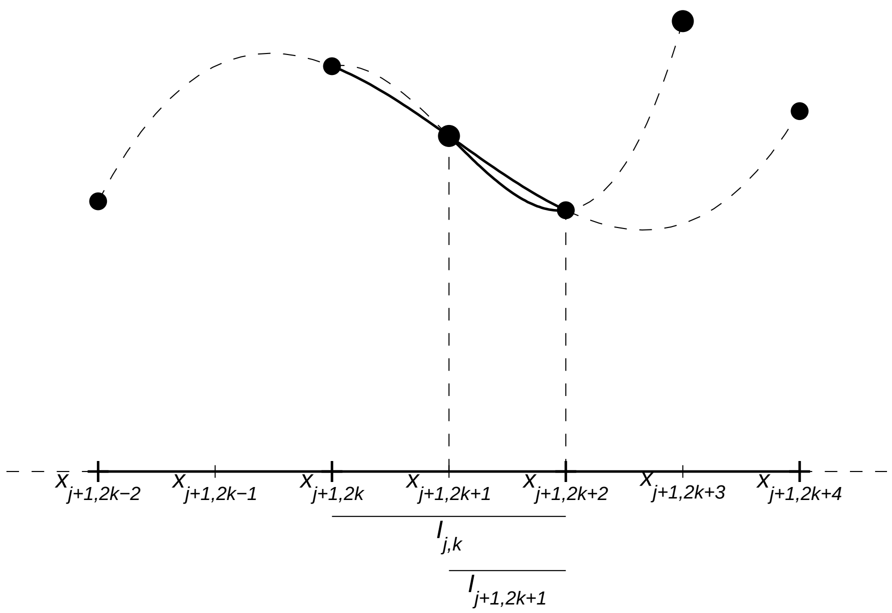

Figure 3.1: $s_{j+1}(x) - s_j(x),\ \ x \in I_{j+1,2k+1}$

$c_1 = [0,1,2,3]g_{j+1,2k} - [0,1,2,3]g_{j,k-1}$, where $[0,1,2,3]g_{j+1,2k}$ and $[0,1,2,3]g_{j,k-1}$ are the third order divided differences over the values $g_{j+1,2k}, g_{j+1,2k+1}, g_{j+1,2k+2}, g_{j+1,2k+3}$ and $g_{j,k-1}, g_{j,k}, g_{j,k+1}, g_{j,k+2}$ and the leading coefficients of $p_{j+1,2k}(x)$ and $p_{j,k-1}(x)$, respectively. Note that $[0,1,2,3]g_{j,k-1} = [0,1,2,4]g_{j+1,2k}$. Hence $c_1 = -h_{j+1,2k+3}[0,1,2,3,4]g_{j+1,2k}$. See Fig 3.1. Secondly, for $x \in I_{j+1,0}$ we have

$$\begin{aligned} s_{j+1}(x) - s_j(x) &= p_{j+1,0}(x) - p_{j,0}(x) \\ &= c_2(x - x_{j+1,0})(x - x_{j+1,1})(x - x_{j+1,2}) \end{aligned}$$

Here $c_2 = [0,0,1,2]g_{j+1,0} - [0,0,1,2]g_{j,0}$ where $[0,0,1,2]g_{j+1,0}$ and $[0,0,1,2]g_{j,0}$ are the leading coefficients of $p_{j+1,0}(x)$ and $p_{j,0}(x)$. Now note that by definition, $g_{j+1,1}$ is a point on the cubic $p_{j,0}$, so we have that $[0,0,1,2]g_{j+1,0} = [0,0,1,2]g_{j,0}$, so $c_2 = 0$. This is illustrated in Fig 3.2. Finally, for the special case when $x \in I_{j+1,1}$ we have the following:

$$\begin{aligned} s_{j+1}(x) - s_j(x) &= p_{j+1,1}(x) - p_{j,0}(x) \\ &= c_3(x - x_{j+1,0})(x - x_{j+1,1})(x - x_{j+1,2}) \end{aligned}$$

Here $c_3 = [0,1,2,3]g_{j+1,0} - [0,0,2,4]g_{j+1,0}$. Now note again by the definition of $g_{j+1,1}$ we have $[0,0,2,4]g_{j+1,0} = [0,1,2,4]g_{j+1,0}$.
Hence $c_3 = -h_{j+1,3}[0,1,2,3,4]g_{j+1,0}$. See Fig 3.3. □

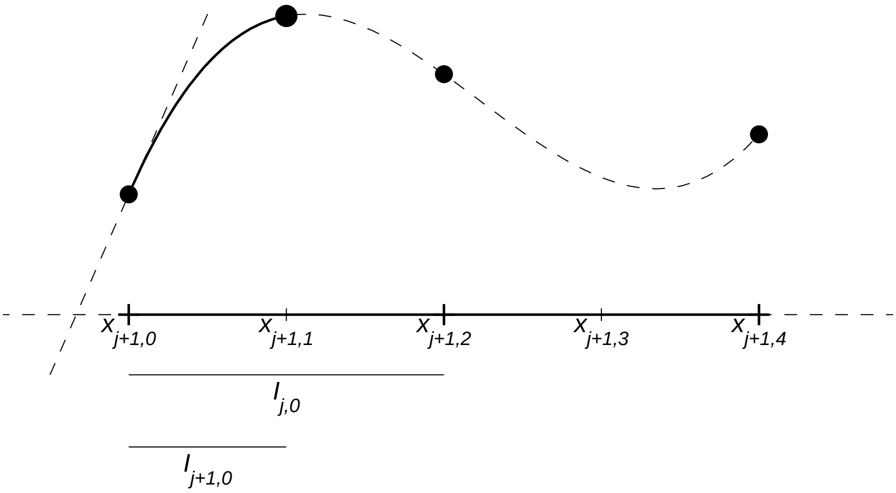

Figure 3.2: $s_{j+1}(x) - s_j(x),\ x \in I_{j+1,0}$

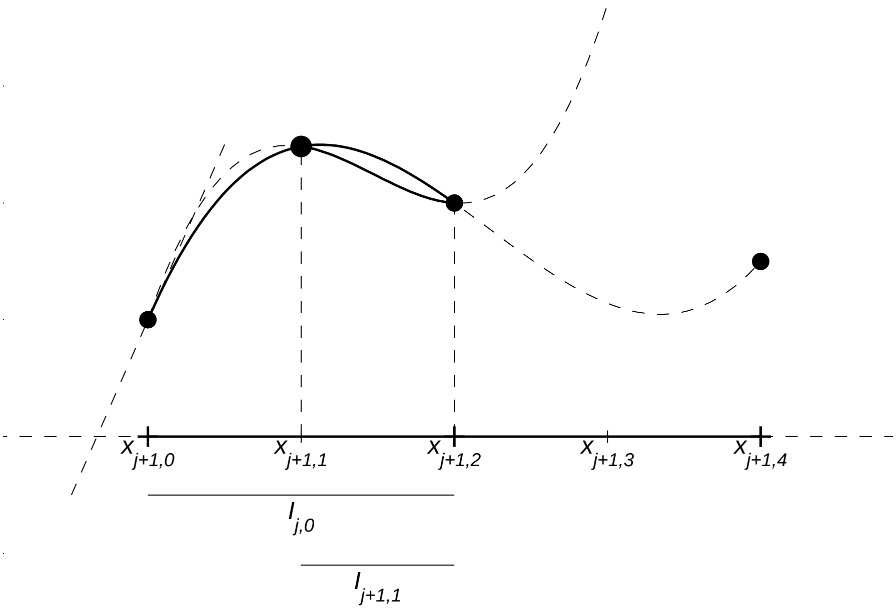

Figure 3.3: $s_{j+1}(x) - s_j(x),\ x \in I_{j+1,1}$

As we see from the lemma below, the difference between two consecutive cubics is expressed through nodal polynomials and fourth order divided differences of the control points on level $j+1$, $g_{j+1,k}$. In order to control the difference we need a bound on the absolute value of the nodal polynomials, and we will eventually need a bound on the derivative values as well.

Lemma 3.2. *For $j \geq 0$ and $x \in I_{j+1,2k}$,*

$$|(\psi^{[2]}_{j+1,2k})^{(r)}(x)| \leq A_r h^{2-r}_{j+1,2k} h_{j,k}, \qquad r = 0, 1, 2,$$

where $A_0 = 1$, $A_1 = 3$, and $A_2 = 6$.

The proof follows directly by differentiation.
After hopefully proving that our scheme produces a C^0 limit function the task of proving C^1 continuity remains. For that analysis need to control the jumps in the first derivatives of the piecewise cubics. Let $s'_{j,-}(x), s'_{j,+}(x)$ denote the left and right derivatives of s_j at x, respectively. s_j is cubic in each interval $I_{j,k}$, so both the left and right derivatives are well-defined.

Lemma 3.3. *For $j \geq 0$ we have*

$$s'_{j,+}(x_{j,0}) - s'_{j,-}(x_{j,0}) = 0 \tag{3.10}$$

$$s'_{j,+}(x_{j,k}) - s'_{j,-}(x_{j,k}) = -h_{j,k-1} h_{j,k} \tilde{g}^{[4]}_{j,k-2} \ \textit{for} \ \ k \geq 1. \tag{3.11}$$

Proof. For $x_{j,k}$ where $k > 1$ the proof is provided in [11], but we need to look at $x_{j,0}$ and $x_{j,1}$ which are the points affected by the new rule. First of all, for $x_{j,0}$, the result follows from Lemma 3.1 since $s_{j,+}(x_{j,0}) = s_{j,-}(x_{j,0}) = p_{j,0}(x_{j,0})$. For $x \in I_{j,0}$ we have:

$$s_j(x) = p_{j,0}(x)$$

By the Newton form; differentiating and setting $x = x_{j,1}$ yields:

$$\begin{aligned} p'_{j,0}(x_{j,1}) = {} & [0,0]g_{j,0} + 2(x_{j,1} - x_{j,0})[0,0,1]g_{j,0} \\ & + [0,0,1,2]g_{j,0}(x_{j,1} - x_{j,0})^2 \end{aligned}$$

For x in $I_{j,1}$:

$$s_j(x) = p_{j,1}(x)$$

Then

$$\begin{aligned} p'_{j,0}(x_{j,1}) = {} & [0,1]g_{j,0} + (x_{j,1} - x_{j,0})[0,1,2]g_{j,0} \\ & + (x_{j,1} - x_{j,0})(x_{j,1} - x_{j,2})[0,1,2,3]g_{j,0} \end{aligned}$$

Setting $s'_{j,-}(x_{j,1}) = p'_{j,0}(x_{j,1})$ and $s'_{j,+}(x_{j,1}) = p'_{j,1}(x_{j,1})$, then

$$s'_{j,+}(x_{j,1}) - s'_{j,-}(x_{j,1}) = -h_{j,0} h_{j,1} \tilde{g}^{[4]}_{j,-1}$$

where $\tilde{g}^{[4]}_{j,-1} = h^{[3]}_{j,0}[0,0,1,2,3]g_{j,0} = [0,1,2,3]g_{j,0} - [0,0,1,2]g_{j,0}$ □

Note that here the differences of third order divided differences appear. We want to find a bound on these differences, in order to do so we derive a rule for these.

Lemma 3.4.

$$\tilde{g}^{[4]}_{j+1,-1} = -\frac{h_{j+1,3}}{h^{[2]}_{j+1,1}}\tilde{g}^{[4]}_{j,-1} \tag{3.12}$$

$$\tilde{g}^{[4]}_{j+1,2k} = \frac{h^{[4]}_{j+1,2k}}{h^{[2]}_{j+1,2k+1}}\tilde{g}^{[4]}_{j,k-1}, \tag{3.13}$$

$$\tilde{g}^{[4]}_{j+1,2k+1} = -\frac{h_{j+1,2k}}{h^{[2]}_{j+1,2k+1}}\tilde{g}^{[4]}_{j,k-1} - \frac{h_{j+1,2k+5}}{h^{[2]}_{j+1,2k+3}}\tilde{g}^{[4]}_{j,k} \tag{3.14}$$

$$\tag{3.15}$$

Proof. For $k \geq 0$, the even case and the odd case k the equation are proven in [11] but for $2k+1 = -1 \Leftrightarrow k = -1$ we have

$$\begin{aligned}
\tilde{g}^{[4]}_{j+1,-1} &= [0,0,1,2,3]\tilde{g}_{j+1,0} \\
&= [0,1,2,3]g_{j+1,0} - [0,0,1,2]g_{j+1,0} \\
&= [0,1,2,3]g_{j+1,0} - [0,0,2,4]g_{j+1,0} \\
&= [0,1,2,3]g_{j+1,0} - [0,1,2,4]g_{j+1,0} \\
&= -h_{j+1,3}[0,1,2,3,4]g_{j+1,0}
\end{aligned}$$

Now note that

$$\begin{aligned}
[0,1,2,3,4]g_{j+1,0} &= \frac{[0,2,3,4]g_{j+1,0} - [0,1,2,4]g_{j+1,0}}{h^{[2]}_{j+1,1}} \\
&= \frac{[0,1,2,3]g_{j,0} - [0,0,1,2]g_{j,0}}{h^{[2]}_{j+1,1}} \\
&= \frac{h^{[3]}_{j,0}}{h^{[2]}_{j+1,1}}[0,0,1,2,3]g_{j,0} \\
&= \frac{\tilde{g}^{[4]}_{j,-1}}{h^{[2]}_{j+1,1}}
\end{aligned}$$

Hence

$$\tilde{g}^{[4]}_{j+1,-1} = -\frac{h_{j+1,3}}{h^{[2]}_{j+1,1}}\tilde{g}^{[4]}_{j,-1},$$

□

The following lemma expresses that if the sequence s_j form a Cauchy sequence, then it converges to a continuous function. The lemma is similar to Lemma 1.1, but also provides the convergence rate.

Lemma 3.5. *Suppose there are constants $C_0 > 0$ and β, $0 < \beta < 1$, such that*

$$\|s_{j+1} - s_j\| \leq C_0 \beta^j, \qquad j \geq 0. \tag{3.16}$$

Then there is a continuous limit function

$$g(x) := \lim_{j \to \infty} s_j(x), \qquad x \in \mathbb{R}, \tag{3.17}$$

and the rate of convergence is $O(\beta^j)$ as $j \to \infty$; specifically,

$$\|g - s_j\| \leq C_0 \beta^j / (1 - \beta). \tag{3.18}$$

Lemma 3.6. *Suppose, in addition to the hypothesis of Lemma 3.5, that there are constants $C_1 > 0$ and γ, $0 < \gamma < 1$, such that*

$$\|s'_{j+1,\pm} - s'_{j,\pm}\| \leq C_1 \gamma^j, \qquad j \geq 0, \tag{3.19}$$

and suppose further that for all grid points $x \in X$,

$$|s'_{j,+}(x) - s'_{j,-}(x)| \to 0 \qquad \text{as } j \to \infty. \tag{3.20}$$

Then the limit function g in (3.17) is continuously differentiable and

$$\|g' - s'_{j,\pm}\| \leq C_1 \gamma^j / (1 - \gamma). \tag{3.21}$$

Proof. The proof of both lemma 3.5 and 3.6 are given in [11] □

With the lemmas above it is clear what conditions must be met. By the rules in Lemma 3.4 we construct the two constants we need, while Lemma 3.3 shows the decay of the first order derivative at the grid points required in (3.20). Now we use that the multi-level grid is assumed to be dyadically balanced.

Lemma 3.7. *Suppose $\lambda < 1$. Then for all $j \geq 0$ and $k \geq -1$,*

$$|\tilde{g}^{[4]}_{j,k}| \leq \frac{C\lambda^j}{h_{j,k+1} h_{j,k+2}},$$

where $C = h^2 M$.

Proof. Define

$$h = \sup_{k \geq 0} h_{0,k} \qquad \text{and} \qquad M := \sup_{k \geq -1} |\tilde{g}^{[4]}_{0,k}|.$$

Let

$$G_{j,k} := h_{j,k+1} h_{j,k+2} \tilde{g}^{[4]}_{j,k}.$$

Then from (3.13–3.14), we obtain a scheme for $G_{j,k}$. For fixed j and k where $k \geq 0$, the lemma was proven in [11], thus we will only provide the argument for $2k+1=-1$ which are where the subdivision rules differ from the standard setting. For $2k+1=-1$ we have:

$$G_{j+1,-1} = -cG_{j,-1}$$

where

$$c = \frac{h_{j+1,0}h_{j+1,1}h_{j+1,3}}{h_{j+1,1}^{[2]}h_{j,0}h_{j,1}}.$$

So for c,

$$\begin{aligned} c &\leq \frac{h_{j+1,0}h_{j+1,3}}{h_{j,0}h_{j,1}} \\ &\leq \frac{h_{j+1,3}}{h_{j,1}}\lambda \\ &\leq \lambda^2 < \lambda, \end{aligned}$$

since $\lambda < 1$. Hence

$$\begin{aligned} |G_{j+1,-1}| &< \lambda |G_{j,-1}| \\ &< \lambda^{j+1}h^2 M. \end{aligned}$$

Adding this to the results from [11] implies that

$$\begin{aligned} \sup_{k\geq -1} |G_{j+1,k}| &\leq \lambda \sup_{k\geq -1} |G_{j,k}| \\ &\leq \lambda^{j+1} \sup_{k\geq -1} |G_{0,k}| \\ &\leq \lambda^{j+1}h^2 M. \end{aligned}$$

□

We are now ready to use all this machinery to prove the following theorem:

Theorem 3.1. *Assume $\lambda < 1$, then the scheme has a limit function $g \in C^1[0,n]$ and, moreover,*

$$\|g - s_j\| \leq h^3 M \lambda^{2j+2}/(1-\lambda^2), \tag{3.22}$$

and

$$\|g' - s'_{j,\pm}\| \leq 3h^2 M \lambda^{j+1}/(1-\lambda). \tag{3.23}$$

Proof. In order to apply Lemmas 3.5 and 3.6, let $x \in (x_{j+1,2k}, x_{j+1,2k+1})$ and consider the first case of (3.9). From equation (3.13) we have

$$|g^{[4]}_{j+1,2k-2}| = \frac{|\tilde{g}^{[4]}_{j,k-2}|}{h^{[2]}_{j+1,2k-1}} \leq \frac{|\tilde{g}^{[4]}_{j,k-2}|}{h_{j+1,2k}}. \tag{3.24}$$

This and Lemma 3.2 then show that

$$|s^{(r)}_{j+1}(x) - s^{(r)}_j(x)| \leq A_r h_{j+1,2k-2} h_{j,k} h^{1-r}_{j+1,2k} |\tilde{g}^{[4]}_{j,k-2}|, \qquad r = 0, 1, 2. \tag{3.25}$$

Therefore, since $h_{j+1,2k-2} \leq \lambda h_{j,k-1}$, Lemma 3.7 implies

$$|s^{(r)}_{j+1}(x) - s^{(r)}_j(x)| \leq A_r h^{1-r}_{j+1,2k} h^2 M \lambda^{j+1}, \qquad r = 0, 1, 2. \tag{3.26}$$

Now, to apply Lemma 3.5, we let $r = 0$ and noting that $h_{j+1,2k} \leq \lambda^{j+1} h$, we have

$$|s_{j+1}(x) - s_j(x)| \leq h^3 M \lambda^{2(j+1)}.$$

The same inequality holds for $x \in I_{j+1,2k+1}$ and so (3.16) holds with $\beta = \lambda^2$ and $C_0 = \lambda^2 h^3 M$, and therefore the scheme converge to a continuous limit function g satisfying (3.22).

Next we want to apply Lemma 3.6. The case $r = 1$ of (3.26) gives

$$|s'_{j+1}(x) - s'_j(x)| \leq 3h^2 M \lambda^{j+1},$$

and so (3.19) holds with $\gamma = \lambda$ and $C_1 = 3\lambda h^2 M$. Further, (3.20) holds because by Lemmas 3.3 and 3.7,

$$|s'_{j,+}(x_{j,k}) - s'_{j,-}(x_{j,k})| \leq h^2 M \lambda^j. \tag{3.27}$$

Thus the criteria for Lemma 3.6 are fulfilled and g is C^1 and satisfies (3.23). □

Now we have proved that our scheme has a limit function $g \in C^1[0, n]$, and it follows that the prescribed derivative is interpolated.

Corollary 3.1 (Prescribed derivative is interpolated)**.** *Let $g \in C^1[0, n]$ be the limit function of the scheme (3.2), and m_0 be the precribed derivative at x_0. Then*

$$g'(x_0) = m_0$$

Proof. The functions $s'_{j,\pm}$ converge uniformly to g' in $[0, 1]$ by Theorem 3.1, and $s'_{j,\pm}(x_0) = m_0 \ \forall j \ \geq 0$ by Lemma 3.3. Since the convergence is uniform, it is also pointwise, hence $g'(x_0) = m_0$. □

In this section we have proven that the subdivision scheme converge to a continuously differentiable function for a dyadically balanced grid. In addition we proved that the limit function has the prescribed derivative.

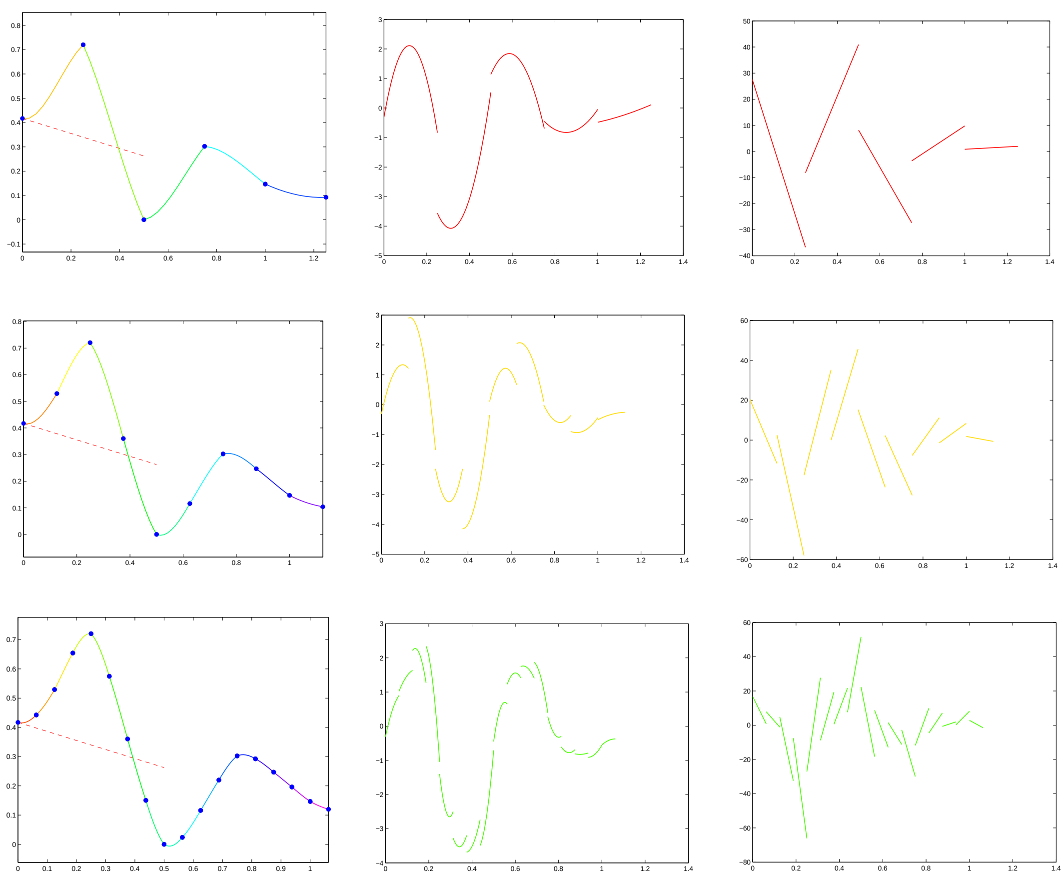

Figure 3.4: Behaviour of the piecewise cubic s_j, piecewise quadratic discontinuous s_j' and the piecewise linear s_j'' after $0, 1, 2$ iterations

3.3 Smoothness

Recall that a n times differentiable, bounded and continuous function $f : [a, b] \to \mathbb{R}$ is said to be Hölder-continuous with exponent $n + \alpha$ with $0 < \alpha < 1$, if there exist a constant $C > 0$ such that

$$\frac{|f^{(n)}(y) - f^{(n)}(x)|}{(y - x)^\alpha} \leq C \text{ for } a \leq x < y \leq b \tag{3.28}$$

For regular multilevel grids the smoothness of the orginary four-point scheme is shown to be $C^{2-\epsilon}$ for any $\epsilon > 0$, [11, 4]. This regularity can be lost if the new odd grid points are placed to far away from the midpoints. In [4] it was conjectured that it the scheme will produce a $C^{2-\epsilon}$ limit function whenever $\lambda < 1$ for a dyadically balanced multi-level grid, while [11] showed that at least for $\lambda \leq 0.7142$ this regularity is maintained. However, the conjecture was proven wrong in [10], where it is shown that one upper bound for λ is 0.8847 by construction of a counterexample. We have no hopes of proving any higher smoothness for our new scheme so to find our Hölder exponent for our we look at the difference

$$|g'(y) - g'(x)|$$

where $x, y \in \mathbb{R}$ with $x < y$ and $y - x \leq h_\star := \inf_k h_{0,k}$. During the refinements, more knots are inserted into the interval (x, y) and we let $n_j(x, y) = \#\{x_{j,k} \mid x < x_{j,k} < y\}$, it is clear that $n_j(x, y) \to \infty$ as $j \to \infty$ and therefore we can find some $l \in \mathbb{N}$, such that $n_l(x, y) \in \{2, 3\}$. Set $r = n_l(x, y)$, then there is some $k \in \mathbb{Z}$ such that

$$x_{l,k} \leq x < x_{l,k+1} < \ldots < x_{l,k+r} < y \leq x_{l,k+r+1}$$

By the triangle inequality we can express the difference in the first derivative.

$$|g'(y) - g'(x)| \leq |g'(y) - s'_{l,-}(y)| + |s'_{l,-}(y) - s'_{l,+}(x)| + |s'_{l,+}(x) - g'(x)| \tag{3.29}$$

The first and the last term is bounded by (3.23), while the middle term needs more investigation. Let $y_0 = x, y_i = x_{l,k+i}$,for $, i = 1, 2, \ldots, r$ and $y_{r+1} = y$. Then,

$$s'_{l,-}(y) - s'_{l,+}(x) = \sum_{i=0}^{r} (s'_{l,-}(y_{i+1}) - s'_{l,+}(x)) + \sum_{i=1}^{r} (s'_{l,+}(y_i) - s'_{l,-}(y_i)) \tag{3.30}$$

$$= \sum_{i=0}^{r} (y_{i+1} - y_i) s''(\xi_i) + \sum_{i=1}^{r} (s_{l,+}(y_i) - s_{l,-}(y_i)), \tag{3.31}$$

for some $\xi_i \in (y_i, y_{i+1})$. Here the need to control the second derivative is evident, and under the assumption that the multilevel grid is dyadically balanced it is proven in [11] that

$$|s''_j(x)| \leq Cj \frac{\lambda^{j-1}}{h_{j,k}} + D$$

when $x \in (x_{j,k}, x_{j,k+1})$, while the bound is tightend if we consider $\lambda < \lambda_0 \approx 0.7142$

$$|s_j''(x)| \leq Cj + D$$

Leading to the estimate

$$\frac{|g'(y) - g'(x)|}{(y-x)^\alpha} \leq (Cj + D)\left(\frac{\lambda}{(1-\lambda)^\alpha}\right)^j$$

for the simply dyadically balanced case, and

$$\frac{|g'(y) - g'(x)|}{(y-x)^\alpha} \leq (Cj + D)\lambda^{j(1-\alpha)} \tag{3.32}$$

when $\lambda < \lambda_0$, proving in both cases that $g \in C^{2-\epsilon}$. A key ingredient for proving (3.32) [11] was to find a constant $C > 0$ s.t

$$\tilde{g}^{[4]}_{j,k} \leq \frac{C}{h^{[2]}_{j,k+1}} \tag{3.33}$$

as mentioned in the first chapter. In [10] it is constructed a family of grids such that (3.33) does not hold. In this chapter we will try to find an upper bound for the Hölder exponent for *the cubic case*. The main idea is again to use the lemmas derived in [11], and we conjecture that the upper bound on the λ will be the same as for the ordinary four-point scheme, since the set of lemmas and theorem derived for our new scheme so far have been quite similar to those of the ordinary four-point scheme. From the cited article [11] we have the following lemma:

Lemma 3.8. *Suppose $\lambda \leq \lambda_0 \approx 0.7142$. Then for all $j \geq 0$ and $k \geq -1$,*

$$|\tilde{g}^{[4]}_{j,k}| \leq \frac{C}{h^{[2]}_{j,k+1}},$$

where $C = hM$.

Proof. Let

$$G_{j,k} := h^{[2]}_{j,k+1}\tilde{g}^{[4]}_{j,k}.$$

From (3.12–3.14) we obtain a scheme for $G_{j,k}$. For fixed j and k,

$$\begin{aligned} G_{j+1,-1} &= -bG_{j,-1}, \\ G_{j+1,2k} &= G_{j,k-1}, \\ G_{j+1,2k+1} &= -aG_{j,k-1} - bG_{j,k}, \end{aligned}$$

where

$$a = \frac{h_{j+1,2k}h^{[2]}_{j+1,2k+2}}{h^{[2]}_{j+1,2k+1}h^{[2]}_{j,k}}, \quad b = \frac{h^{[2]}_{j+1,2k+2}h_{j+1,2k+5}}{h^{[2]}_{j+1,2k+3}h^{[2]}_{j,k+1}},$$

We work with induction on j, so we assume that for all $k \geq -1$

$$|\tilde{g}^{[4]}_{j,k}| \leq \frac{C}{h^{[2]}_{j,k+1}}.$$

In [11] it was shown that $a+b \leq 1$ whenever $\lambda \leq \lambda_0$, hence the lemma hold for $k > -1$. So in order to prove the lemma for $k = -1$, we need to prove is that b is bounded by 1, for some upper bound on λ. So we look at

$$G_{j+1,-1} = -bG_{j,-1}$$

where

$$b = \frac{h^{[2]}_{j+1,0}h_{j+1,3}}{h^{[2]}_{j+1,1}h^{[2]}_{j,0}}$$

and by [11] we know that

$$b = b(\mu) \leq \frac{\lambda}{(\sqrt{1-\lambda}+\sqrt{1-\mu})^2}$$

where $\mu = \frac{h_{j+1,0}}{h_{j,0}}$. b is a convex function in μ since the second derivative is non-negative and the maximum is therefore attained at either $(1-\lambda)$ or λ. We see that

$$b(\mu) \leq b(\lambda) = \frac{\lambda}{4(1-\lambda)},$$

which is less than or equal to 1 whenever $\lambda \leq 0.8$ and this obviously holds since $0.8 > 0.7142$. Then we can conclude that

$$\begin{aligned} |G_{j+1,-1}| &= |b||G_{j,-1}| && (3.34)\\ &\leq |G_{j,-1}| && (3.35)\\ &\leq C && (3.36) \end{aligned}$$

by assumption. Hence

$$h^{[2]}_{j+1,0}|\tilde{g}^{[4]}_{j+1,-1}| \leq C, \tag{3.37}$$

which proves the lemma for the special case where $k = -1$. $\square$

The overall goal is to show that the derivative g' is Hölder continuous with exponent $1-\epsilon$ under the assumption that λ is in the range $\lambda \leq \lambda_0$. The following lemmas taken directly from [11] provides the necessary pieces of the puzzle.

Lemma 3.9. *If* $\lambda \leq \lambda_0$ *then*

$$|s'_{j+1}(x) - s'_j(x)| \leq Ch_{j,k}, \qquad x_{j,k} < x < x_{j,k+1}, \tag{3.38}$$

and

$$|s'_{j+1}(x) - s'_j(x)| \leq Ch_{j+1,2k-2}, \qquad x_{j+1,2k} < x < x_{j+1,2k+1}. \tag{3.39}$$

Lemma 3.10. *If $\lambda \leq \lambda_0$ then*

$$|g'(x) - s_j'(x)| \leq C h_{j,k}, \qquad x_{j,k} < x < x_{j,k+1}. \tag{3.40}$$

Lemma 3.11. *If $\lambda \leq \lambda_0$ then*

$$|g'(x) - s_j'(x)| \leq C(x - x_{j,k-1}), \qquad x_{j,k} < x < x_{j,k+1}. \tag{3.41}$$

We are now ready to prove following theorem.

Theorem 3.2. *If $\lambda \leq \lambda_0$, the function g' is Hölder continuous with exponent $1-\epsilon$ for any small $\epsilon > 0$.*

Proof. We return to the triangle inequality (3.29):

$$|g'(y) - g'(x)| \leq |g'(y) - s_{l,-}'(y)| + |s_{l,-}'(y) - s_{l,+}'(x)| + |s_{l,+}'(x) - g'(x)|$$

and the expression for the middle term, (3.30). From Lemma 3.8, using (3.10), we have for $j \geq 0$,

$$|s_{j,+}'(x_{j,k}) - s_{j,-}'(x_{j,k})| \leq C \frac{h_{j,k-1} h_{j,k}}{h_{j,k-1}^{[2]}} \leq C \min\{h_{j,k-1}, h_{j,k}\}, \tag{3.42}$$

and it follows that

$$|s_{j,+}'(y_i) - s_{j,-}'(y_i)| \leq C(y-x)$$

in (3.30). Further, applying Lemma 3.8 to the case $r = 2$ of (3.25) gives

$$|s_{j+1}''(x) - s_j''(x)| \leq C,$$

and therefore

$$|s_j''(x)| \leq Cj + D.$$

Applying these two estimates to (3.30), shows that

$$|s_{j,-}'(y) - s_{j,+}'(x)| \leq (Cj + D)(y - x).$$

Next, observe that Lemma 3.11 shows that

$$|g'(y) - s_{j,-}'(y)| \leq C(y - x_{j,k+r-1}) \leq C(y-x)$$

in (3.29), and a similar argument shows that

$$|s_{j,+}'(x) - g'(x)| \leq C(y-x).$$

Then

$$|g'(y) - g'(x)| \leq (Cj + D)(y - x).$$

Since

$$y - x \leq x_{j,k+r+1} - x_{j,k} \leq (r+1)\lambda^j h,$$

this means that

$$\frac{|g'(y) - g'(x)|}{(y-x)^\alpha} \leq (Cj + D)\lambda^{j(1-\alpha)},$$

the right hand side of which is bounded as a function of $j \geq 0$ for any $\alpha \in (0,1)$. $\square$

We can hence conclude that the limit function of *the cubic case* over a dyadically balanced grid g has a Hölder regularity of $C^{2-\epsilon}$, for all values of λ in the range $[0.5, \lambda_0]$. In the proof of Lemma 3.9 we see that we can bound the first difference of fourth order divided differences with a bigger $\lambda, \lambda = 0.8$, so the question remains if $\lambda \leq 0.7142$ can be improved somehow globally.

3.4 Smoothness results for the tensor product extension

In this section we hope to prove that the limit surface of the subdivision scheme that we defined previously in 2.4 is $C^{1,1}$. We will use what we have proven in the previous sections, that the univariate scheme is C^1.

Theorem 3.3. *The limit function* $g : [x_0, x_n] \times [y_0, y_m] \mapsto \mathbb{R}$ *of the tensor product extension scheme defined in 2.4 is* $C^{1,1}$*, i.e* $g \in C^0$, $\frac{\partial g}{\partial x} \in C^0$, $\frac{\partial g}{\partial y} \in C^0$ *and* $\frac{\partial^2 g}{\partial x \partial y} \in C^0$

Proof. It it easy to verify that the limit function (3.43) solves the interpolation problem by insertion of the function values, and by differentiation we can prove that is has the precribed partial and mixed derivatives on the egdes. Differentiation of g also shows that $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial x}$ and $\frac{\partial^2 g}{\partial x \partial y}$ are all in C^0, since they are all linear combinations of basis function and/or derivatives of basis functions with the known smoothness of C^1 and C^0, respectively. Hence $g \in C^{1,1}$.

$$\begin{aligned}
g(x,y) = &\sum_{k=0}^{n}\sum_{j=0}^{m} c_{k,l}\phi_k^0(x)\phi_l^1(y) \\
&+ \sum_{k=0}^{n} d_k \phi_k^0(x)\psi_0^1(y) \\
&+ \sum_{k=0}^{n} e_k \phi_k^0(x)\psi_m^1(y) \\
&+ \sum_{l=0}^{m} h_l \phi_l^1(y)\psi_0^0(x) \\
&+ \sum_{l=0}^{m} v_l \phi_k^1(y)\psi_n^0(x) \\
&+ a_0\psi_0^0(x)\psi_0^1(y) \\
&+ a_1\psi_0^0(x)\psi_m^1(y) \\
&+ a_2\psi_n^0(x)\psi_0^1(y) \\
&+ a_3\psi_n^0(x)\psi_m^1(y).
\end{aligned} \tag{3.43}$$

□

3.5 Approximation order of the cubic case

In this section we will try to uncover the approximation order of the scheme on a dyadically balanced grid by comparing the limit function to a given function with a bounded fourth order derivative. Recall from the previous section that we defined $h = \sup_k h_{0,k}$.

Theorem 3.4. *If $\lambda < 1$ and f, the function sampled, has a bounded fourth order derivative in $\mathbb{R}$, there exists constants C_0, C_1, only dependent on λ such that*

$$\|f - g\| \leq C_0 h^4 \|f^{(4)}\| \tag{3.44}$$

$$\tag{3.45}$$

Proof. Using the triangle inequality we have that

$$|f(x) - g(x)| \leq |f(x) - s_0(x)| + |s_0(x) - g(x)| \tag{3.46}$$

This inequality is bounded in [11] for the four-point scheme, where we consider an arbitrary point $x \in (x_{0,k}, x_{0,k+1})$ for some k, and find the bound to be

$$\|f(x) - g(x)\| \leq \left(\frac{9}{384} + \frac{\lambda^2}{6(1-\lambda^2)}\right) h^4 \|f^{(4)}\|$$

As noted before $s_0(x)$ is only altered in the first interval $[x_{0,k}, x_{0,k+1}]$, so we need to find a bound for difference this interval. Let $x \in (x_{0,k}, x_{0,k+1})$ and let us consider the first term of (3.46) first. Recall that

$$s_0(x) = [0]g_{0,0} + (x - x_{0,0})[0,0]g_{0,0} + (x - x_{0,0})^2[0,0,1]g_{0,0} + (x - x_{0,0})^2(x - x_{0,1})[0,0,1,2]g_{0,0}$$

Newtons error formula applies to osculatory interpolation problems as well due to the continuity of the divided differences, so by for example eq. 4.3.8 [2] page. 383 we have

$$f(x) - s_0(x) = \frac{f^{(4)}(\xi_x)}{4!}(x - x_{0,0})^2(x - x_{0,1})(x - x_{0,2}), \tag{3.47}$$

where $\xi_x \in (x_{0,0}, x_{0,2})$. Further we find that

$$|x - x_{0,0}||x - x_{0,1}| \leq \frac{h^2}{4} \tag{3.48}$$

while,

$$|x - x_{0,0}||x - x_{0,2}| \leq h^2 \tag{3.49}$$

Leading to

$$|f(x) - s_0(x)| \leq \frac{|f^{(4)}(\xi_x)|}{96} h^4, \tag{3.50}$$

for $\xi_x \in (x_{0,0}, x_{0,2})$. For the second term the estimate is similar to the one in [11]

$$|s_0(x) - g(x)| \leq h^3 M \frac{\lambda^2}{(1-\lambda^2)}, \tag{3.51}$$

by (3.22), further we know that

$$\begin{aligned} M &= \sup_k \{ \tilde{g}^{[4]}_{0,k} \} && (3.52)\\ &= \max\{h^{[3]}_{0,0} g^{[4]}_{0,-1}, \sup_{k\geq 0}\{h^{[4]}_{0,k} g^{[4]}_{0,k}\}\} && (3.53)\\ &\leq 4h \sup_{k\geq 0} |g^{[4]}_{0,k}| && (3.54)\\ &\leq 4h \frac{\|f^{(4)}\|}{4!} && (3.55)\\ &\leq h \frac{\|f^{(4)}\|}{6} && (3.56) \end{aligned}$$

Combining (3.51) and (3.56) we get

$$|s_0(x) - g(x)| \leq \frac{h^4 \|f^{(4)}\| \lambda^2}{6(1-\lambda^2)} \tag{3.57}$$

Then

$$\|f(x) - g(x)\| \leq \left(\max\{\frac{1}{96}, \frac{9}{384}\} + \frac{\lambda^2}{6(1-\lambda^2)} \right) h^4 \|f^{(4)}\| \tag{3.58}$$

Note that $\frac{1}{96} < \frac{9}{384}$ so we use the old value as our bound, thus the theorem holds with $C_0 = \frac{9}{384} + \frac{\lambda^2}{6(1-\lambda^2)}$. □

4 Conclusions

4.1 Summary and future work

In this thesis we have introduced a family of univariate interpolatory subdivision scheme for interpolating function values and derivative values at the end points. For a special member of this family, *the cubic case* we proved that the limit function is continuously differentiable, and even Hölder-continuous with exponent $2-\epsilon$ for any $\epsilon > 0$ under a certain restriction on the parametrisation. An extension of *the cubic case* to a tensor product scheme was provided, and shown to inherit smoothness properties from the univariate case, as we claimed that the limit surface was $C^{1,1}$.

As mentioned earlier in this thesis, an interesting question when it comes to the schemes introduced in the second chapter could be to determine if they have any of the properties that the Dubuc-Deslauriers scheme have. Are the schemes we introduced convergent in general, and is the smoothness of these schemes somehow connected with the smoothness of the Dubuc-Deslauriers schemes? When the number of derivatives are elevated, the degree also becomes higher and higher, and we are not sure of the applicability of the schemes. It might seem rather unnatural to specify values for a high order degrees of the end point derivatives. Due to the visual weakness we commented on in our implementation of the bicubic tensor product scheme, it is also natural to investigate if the visual appearance and smoothness can be enhanced somehow if we define the scheme differently or base the tensor product extension on a higher order scheme. A suggestion is to provide a second derivative and a first derivative at the first point giving a scheme where $s = 2, d = 5$ in the univariate case and extend this to a tensor product scheme. In the univariate case we would use the Dubuc-Deslauriers scheme of order 5 everywhere, except for calculation the first and second new odd points. The selection of the new odd points is illustrated in Fig 4.1. In our notations from the previous chapters this quintic scheme yields

$$\begin{aligned}
f_{j+1,0} &= f_{j,0}, \\
f_{j+1,1} &= p^{[5]}_{j,0}(x_{j+1,1}), \\
f_{j+1,2} &= f_{j,1}, \\
f_{j+1,3} &= p^{[5]}_{j,1}(x_{j+1,3}), \\
f_{j+1,2k} &= f_{j,k}, \\
f_{j+1,2k+1} &= p^{[5]}_{j,k}(x_{j+1,2k+1}), k > 2,
\end{aligned}$$

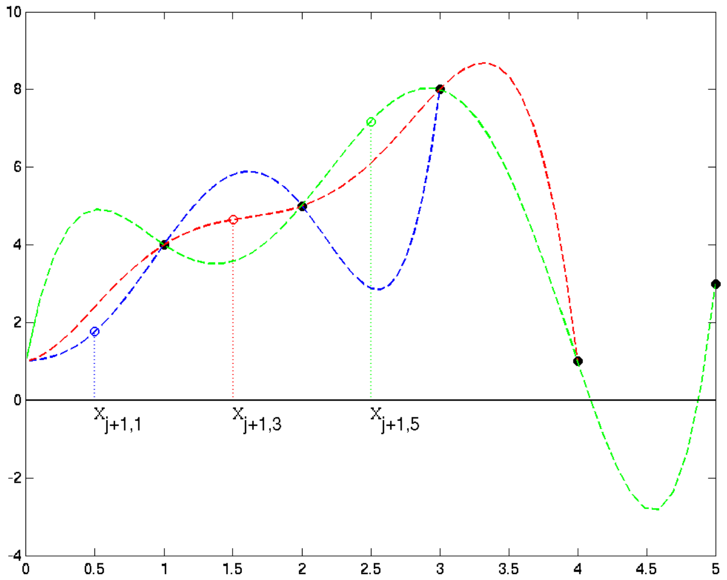

Figure 4.1: Selection of new points in the quintic case

where $p^{[5]}_{j,0}(x)$ is the quintic osculatory interpolant to $m^2_0, m^1_0, f_{j,0}, f_{j,1}, f_{j,2}, f_{j,3}$, and $p^{[5]}_{j,1}(x)$ is the quintic osculatory interpolant to $m^1_0, f_{j,0}, f_{j,1}, f_{j,2}, f_{j,3}, f_{j,4}$, while as usual $p_{j,k}(x)$ is the Lagrange interpolant to $f_{j,k-2}, f_{j,k-1}, f_{j,k}, f_{j,k+1}, f_{j,k+2}, f_{j,k+3}$. As we did for the cubic case, we generate similar rules for the two last odd points as well. We predict a better visual appearance if we use a tensor product extension of this scheme, instead of the cubic scheme, since the normal Dubuc-Deslauriers scheme of order 5 is known to produce C^2 curves. We conjecture that this modified quintic scheme will be C^2 as well, but a thorough smoothness analysis is needed. The piecewise polynomial approach used here and developed in [11] may be used as an analysis tool for this case also.

A similar approach as done in the tensor product extension(see section 2.4) of the cubic case was used to define a tensor product biquintic scheme. The notations for introducing this scheme are involved and of little illustrational interest, at least as a part of the conclusion. Out of curiousity, however, we implemented the scheme. Solely based the result we see in the figures 4.4 and 4.5, the visual appearance is enhanced, i.e the join is smoother, by our subjective opinion. The illustration in Fig 4.3 also indicated that the prescribed first derivative is interpolated. We emphasise that no smoothness analysis is done, neither in the univariate case nor the bivariate tensor product case, and we suggest this as a topic for further research.

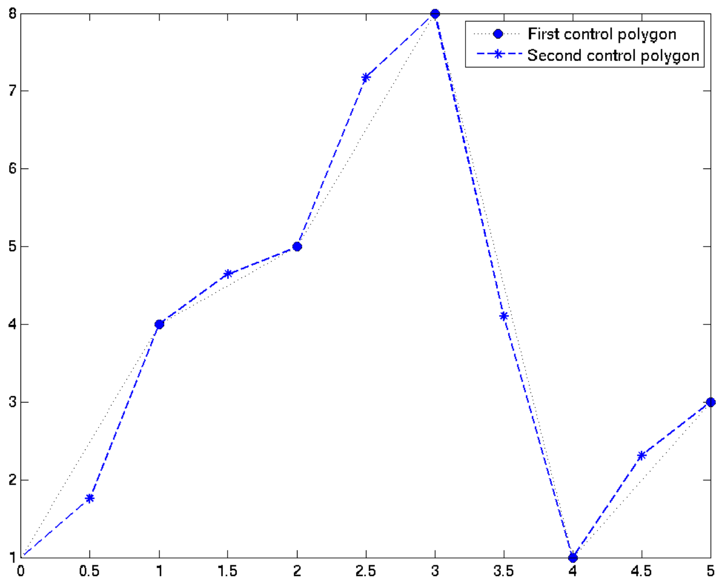

Figure 4.2: The first and second control polygon for the quintic case

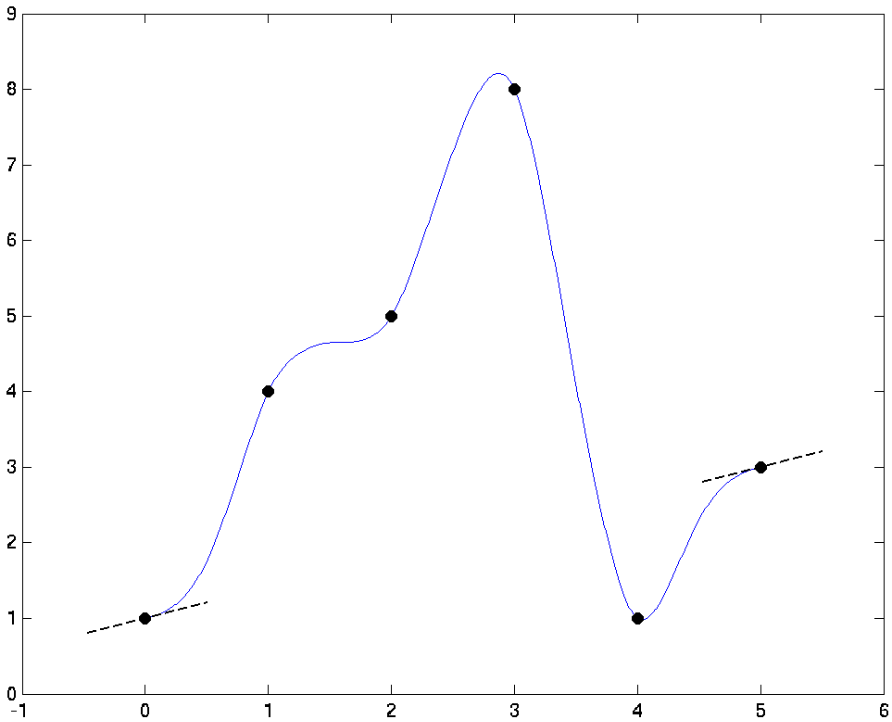

Figure 4.3: An illustration of the limit function of the univariate quintic case

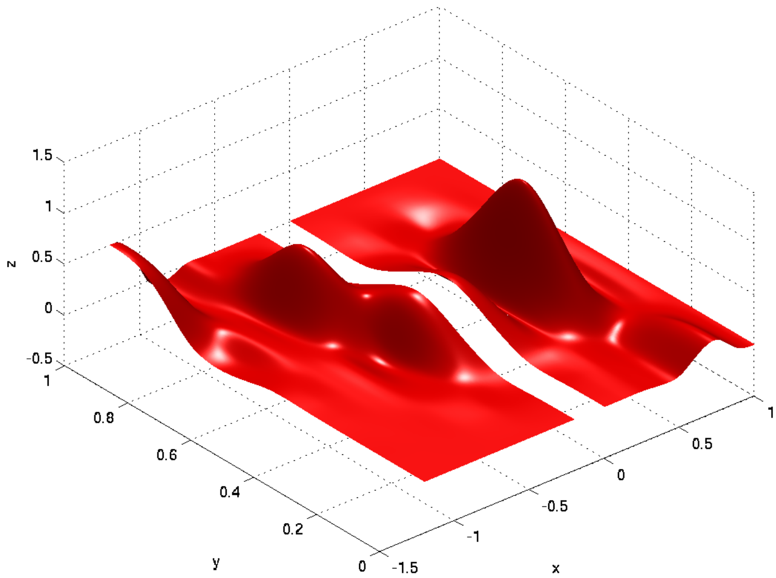

Figure 4.4: Two surface generated by the quintic tensor product scheme.

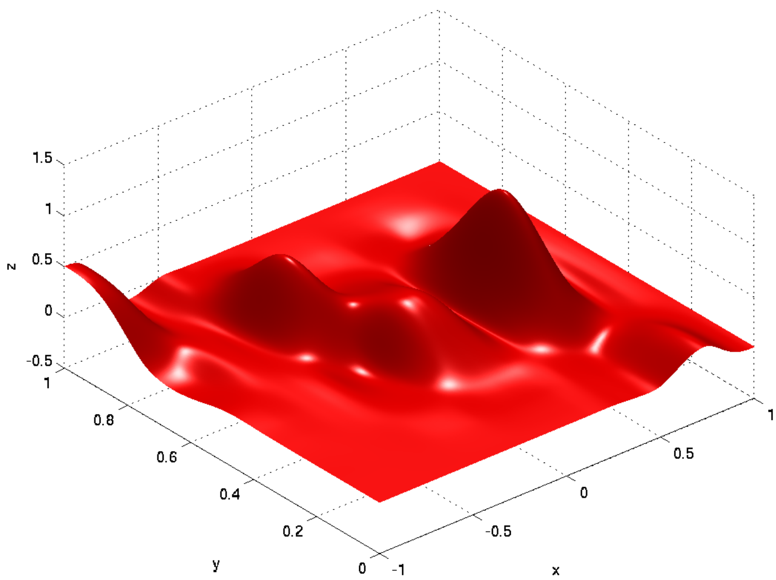

Figure 4.5: The two surfaces in 4.4 joined.

Bibliography

[1] George Merrill Chaikin. An algorithm for high-speed curve generation. *Computer Graphics and Image Processing*, 3(4):346 – 349, 1974.

[2] Germund Dahlquist and Åke Björck. *Numerical Methods in Scientific Computing: Volume 1.* Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2008.

[3] I. Daubechies, I. Guskov, and W. Sweldens. Commutation for irregular subdivision. *Constr. Approx.*, 15(3):381–426, 2001.

[4] Ingrid Daubechies, Igor Guskov, and Wim Sweldens. Regularity Of Irregular Subdivision, 1998.

[5] Gilles Deslauriers and Serge Dubuc. Symmetric iterative interpolation processes. *Constructive Approximation*, 5:49–68, 1989.

[6] Serge Dubuc. Interpolation through an iterative scheme. *Journal of Mathematical Analysis and Applications*, 114(1):185 – 204, 1986.

[7] N Dyn. Subdivision schemes in computer-aided geometric design.

[8] Nira Dyn and David Levin. Subdivision schemes in geometric modelling, 2002.

[9] Nira Dyn, David Levin, and John A. Gregory. A 4-point interpolatory subdivision scheme for curve design. *Computer Aided Geometric Design*, 4(4):257 – 268, 1987.

[10] Michael S. Floater. The loss of Hölder regularity of four-point interpolatory subdivision on irregularly spaced points. *Jaen Journal on Approximation*, 2011.

[11] Michael S. Floater. A piecewise polynomial approach to analyzing interpolatory subdivision. *J. Approx. Theory*, 163(11):1547–1563, November 2011.

[12] E. Isaacson and H.B. Keller. *Analysis of Numerical Methods.* Dover Books on Mathematics. Dover Publications, 1994.

[13] B. M J. Hechler. C1-continuity of the generalized four-point scheme. *Linear Algebra and its Applications*, 430(11?12):3019 – 3029, 2009.

[14] Adi Levin. *Combined Subdivision Schemes.* PhD thesis, Tel-Aviv University, 2000.

[15] Voichita Maxim and Marie-Laurence Mazure. Subdivision schemes and irregular grids. *Numerical Algorithms*, 35(1):1–28.

[16] James R. Munkres. *Topology*. Geometry and Computing. Pearson, 2000.

[17] Pixar. An open source subdivision library. `http://graphics.pixar.com/opensubdiv/`.

[18] Pixar. A short film using subdivision. `http://www.pixar.com/short_films/Theatrical-Shorts`, 1997.

[19] M. Sabin. *Analysis and Design of Univariate Subdivision Schemes*. Geometry and Computing. Springer-Verlag Berlin Heidelberg, 2010.

[20] M. Schatzman. *Numerical Analysis: An Mathematical Introduction*. Clarendon Press, 2002.

[21] I.J Schoenberg. On Hermite-Birkhoff Interpolation. *Journal of Mathematical Analysis and Applications*, 16(3):538 – 543, 1966.

[22] Peter Schr˙Subdivision for modeling and animation. Technical report, SIGGRAPH 1999.

[23] Wim Sweldens. The lifting scheme: A construction of second generation wavelets, 1997.

[24] J. Warren. Binary subdivision schemes for functions over irregular knot sequences. In *Mathematical Methods in CAGD III*, pages 543–562. Vanderbilt University Press, 1995.

[25] Joe Warren and Henrik Weimer. *Subdivision Methods for Geometric Design: A Constructive Approach*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1st edition, 2001.

[26] Cai Zhijie. Modified four-point scheme and its application. *Computer Aided Geometric Design*, 15(3):251 – 260, 1998.

[27] Denis Zorin, Peter Schröder, and Wim Sweldens. Interpolating subdivision for meshes with arbitrary topology. In *Proceedings of the 23rd annual conference on Computer graphics and interactive techniques*, SIGGRAPH '96, pages 189–192, New York, NY, USA, 1996. ACM.